Math–501. Second Midterm. Wednesday, October 31, 2007.

1. State the dominated convergence theorem.
2. State the Jensen inequality.
3. Write the definition of an (absolutely) continuous $\mathbb{R}^d$–valued random vector.
4. Let $1 < p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Let f, g, h be two measurable functions such that $f \in L_{p_1}$, $g \in L_{p_2}$ and $g \in L_{p_3}$. Show that

$$\int |fgh|\, d\mu \leq \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3}.$$

5. Let X have the probability density function

$$f(x) = \begin{cases} \frac{3x^2}{k^3}, & \text{if } \ 0 \leq x \leq k, \\ 0 & \text{else.} \end{cases}$$

   The variance of X is 4. Find k.

6. Let $f(x, y) = 6x$, $0 < x < y < 1$, and zero otherwise. Are X and Y independent? Why?
7. Let X and Y be continuous random variables with joint density function

$$f(x, y) = \begin{cases} \frac{8}{3}xy & \text{for } 0 \leq x \leq 1, x \leq y \leq 2x, \\ 0 & \text{otherwise.} \end{cases}$$

   Calculate the covariance of X and Y.

8. Suppose that X and Y are two independent r.v.'s. Show that if X has an absolutely continuous distribution so does $X + Y$.