Math 501. 1st Homework. Due Monday, September 10, 2007

Homework on "Chapter 1"

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $l \in \mathbb{R}$. Show that $\limsup_{n\to\infty} x_n = l$ if and only if:

   (i) for each $\epsilon > 0$, there exists a positive integer n_0 such that for each $n \geq n_0$ $x_n < l + \epsilon$.

   (ii) for each $\epsilon > 0$ and each positive integer n_0 there exists a positive integer $n \geq n_0$ such that $x_n > l + \epsilon$.

2. Prove that for a sequence of subsets $\{A_n\}_{n=1}^{\infty}$ of the universal set Ω, $\lim_{n\to\infty} A_n$ exists if and only if for each $\omega \in \Omega$, $\lim_{n\to\infty} I(\omega \in A_n)$ exists.

   Hint: Recall that $I(\omega \in A_n)$ is the indicator function of the set A_n.

3. Let $A := \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 1, 0 \leq y < 1\}$. Show that A is neither an open set nor a closed set of $\mathbb{R}^2$, but $A \in \mathcal{B}(\mathbb{R}^2)$.

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A_1, \ldots, A_n \in \mathcal{F}$, where $n \geq 2$. Prove that

$$\mathbb{P}[\cup_{i=1}^{n} A_i] \geq \sum_{i=1}^{n} \mathbb{P}[A_i] - \sum_{1\leq i_1 < i_2 \leq n}^{n} \mathbb{P}[A_{i_1} \cap A_{i_2}].$$

5. Let $A_1, \ldots, A_n$ be n arbitrary events. Show that the probability that exactly one of these n events will occur is

$$\sum_{i=1}^{n} \mathbb{P}(A_i) - 2\sum_{1\leq i_1<i_2\leq n} \mathbb{P}(A_{i_1} \cap A_{i_2}) + 3\sum_{1\leq i_1<i_2<i_3\leq n} \mathbb{P}(A_{i_2} \cap A_{i_2} \cap A_{i_3})$$
$$+ \cdots + (-1)^{n+1} n\mathbb{P}(A_1 \cap A_2 \cdots \cap A_n)$$

   Hint: If $n = 2$, the event consisting of the outcomes where exactly one of the events A_1, A_2 happens is $(A_1 \cup A_2) - (A_1 \cap A_2) = (A_1 - A_2) \cup (A_2 - A_1)$. Its probability is

$$\mathbb{P}(A_1 - A_2) + \mathbb{P}[A_2 - A_1] = \mathbb{P}[A_1] + \mathbb{P}[A_2] - 2\mathbb{P}[A_1 \cap A_2].$$

6. A survey indicates that 80 % of college students from New York state owns a cell phone, 50 % owns an Ipod and 40 % owns both a cell phone and an Ipod. Calculate the probability that a college student from New York state chosen at random owns either a cell phone or an Ipod.

7. The National Institutes of Health has the following information on retired American persons:
(i) 34% have heart disease;
(ii) 23% have cancer;
(iii) 29% have diabetes;
(iv) 12% have heart disease and cancer;
(v) 4% have heart disease and diabetes;
(vi) 8% have cancer and diabetes;
(vii) 1% have heart disease, cancer and diabetes.
Calculate the percentage of retired American persons which do not have neither heart disease, nor cancer, nor diabetes.

8. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(x) & \text{if } x \leq -1, \\ \frac{1}{x} & \text{if } -1 < x < 0, \\ \frac{x^2+4}{x+8} & \text{if } 0 \leq x \leq 2, \\ e^x + \log(x) & \text{if } 2 < x. \end{cases}$$

Show that $f(X)$ is a random variable.

9. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A_1, \ldots, A_n \in \mathcal{F}$ be independent events. Show that

$$\mathbb{P}[\cup_{i=1}^n A_i] = 1 - \prod_{i=1}^n (1 - \mathbb{P}[A_i]).$$

10. Let A and B be two independent events with $\mathbb{P}[A] = 2\mathbb{P}[B]$ and $\mathbb{P}[A \cup B] = \frac{7}{25}$. Find $\mathbb{P}[A]$ and $\mathbb{P}[B]$.