Math 501. 2–nd Homework. Due Wednesday, September 26, 2007.

Homework on "Chapter 3".

Definition 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be collections of measurable subsets of Ω. We say that $\mathcal{C}_1, \dots, \mathcal{C}_m$ are independent, if for each $C_1 \in \mathcal{C}_1, \dots, C_m \in \mathcal{C}_m$, $C_1, \dots, C_m$ are independent events.*

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be independent collections of measurable subsets of Ω. Suppose that $\mathcal{C}_i$ is a π–class for each $1 \le i \le m$. Let $1 \le k < m$. Show that $\sigma(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k)$ and $\sigma(\mathcal{C}_{k+1} \cup \cdots \cup \mathcal{C}_m)$ are independent collections of events.

2. Let X, Y and Z be three independent r.v.'s. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a Borel function. Show that X and $h(Y, Z)$ are independent r.v.'s.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be independent collections of measurable subsets of Ω. Suppose that $\mathcal{C}_i$ is a π–class for each $1 \le i \le m$. Let $1 \le k < l < m$. Show that $\sigma(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k)$, $\sigma(\mathcal{C}_{k+1} \cup \cdots \cup \mathcal{C}_l)$ and $\sigma(\mathcal{C}_{l+1} \cup \cdots \cup \mathcal{C}_m)$ are independent collections of events.

4. Let $\mathcal{C}$ be a class of sets of a universal set Ω. Suppose that $\mathcal{C}$ is a π—class and a λ–class. Show that $\mathcal{C}$ is a σ–field.

5. Let μ_1 and μ_2 be two measures on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Suppose that for each bounded closed triangle T of $\mathbb{R}^2$, $\mu_1(T) = \mu_2(T) < \infty$. Show that $\mu_1 = \mu_2$, i.e. $\mu_1(A) = \mu_2(A)$, for each $A \in \mathcal{B}(\mathbb{R}^2)$.

6. Let A be a Borel set of $\mathbb{R}^d$ and let $a \in \mathbb{R}^d$. Let m be the Lebesgue measure in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. We define

   $$a + A = \{x \in \mathbb{R}^d : \text{there exists } b \in A, \text{such that } x = a + b\}.$$

   (i) Prove that $a + A$ is a Borel set of $\mathbb{R}^d$.
   (ii) Prove that $m(A) = m(a + A)$.

7. Let A and B be two Borel sets of $\mathbb{R}^d$. We define

   $$A + B = \{x \in \mathbb{R}^d : \text{there exists } y \in A, z \in B, \text{such that } x = y + z\}.$$

   Prove that $A + B$ is a Borel set of $\mathbb{R}^d$.

8. Let $X_1, \dots, X_n$ be independent identically distributed r.v.'s. Show that for each set $A \in \mathcal{B}(\mathbb{R}^n)$ and each permutation σ of $\{1, \dots, n\}$,

$$\mathbb{P}\{(X_1, \dots, X_n) \in A\} = \mathbb{P}\{(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \in A\}.$$

9. Let $X_1, \dots, X_n$ be r.v.'s. Suppose that for each set $A \in \mathcal{B}(\mathbb{R}^n)$ and each permutation σ of $\{1, \dots, n\}$,

$$\mathbb{P}\{(X_1, \dots, X_n) \in A\} = \mathbb{P}\{(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \in A\}.$$

   (i) Show that $X_1, \dots, X_n$ are identically distributed r.v.'s.
   (ii) Find r.v.'s $X_1, \dots, X_n$ which are not independent and for each set $A \in \mathcal{B}(\mathbb{R}^n)$ and each permutation σ of $\{1, \dots, n\}$,

$$\mathbb{P}\{(X_1, \dots, X_n) \in A\} = \mathbb{P}\{(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \in A\}$$

10. Given a probability measure P on a measurable space $(\Omega, \mathcal{F})$, the outer measure P^* of P is defined as

$$P^*(E) = \inf\{P(A) : E \subset A, A \in \mathcal{F}\}, E \subset \Omega.$$

   (i) Show that P^* is an outer measure, i.e. it satisfies the conditions in Definition 3.4.1.
   (ii) Show that for each set $E \subset \Omega$, there exists a set $A \in \mathcal{F}$ such that $E \subset A$ and $P^*(E) = P(A)$.