Math 501. 3–rd Homework. Due Monday, October 15, 2007.

Homework on the "Lebesgue Measure".

Name: ....

1. (Extended dominated convergence theorem) Let $\{f_n\}_{n=1}^{\infty}$ and let $\{g_n\}_{n=1}^{\infty}$ be two sequence of measurable functions on the measure space $(\Omega, \mathcal{F}, \mu)$. Suppose that:
   (i) For each $\omega \in \Omega$, $|f_n(\omega)| \le g_n(\omega)$.
   (ii) For each $\omega \in \Omega$, $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.
   (iii) For each $\omega \in \Omega$, $\lim_{n\to\infty} g_n(\omega) = g(\omega)$.
   (iv) $\lim_{n\to\infty} \int_\Omega g_n(\omega)\, d\mu(\omega) = \int_\Omega g(\omega)\, d\mu(\omega)$.
   (v) $\int_\Omega g(\omega)\, d\mu(\omega) < \infty$.
   Show that $\lim_{n\to\infty} \int_\Omega f_n(\omega)\, d\mu(\omega) = \int_\Omega f(\omega)\, d\mu(\omega)$.

2. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

   Show that f is continuous in $[0,1]$ and differentiable in $(0,1)$. Show that $\int_0^1 f'(x)\, dx$ does not exists as a Lebesgue integral.

3. Let

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & \text{if}(x,y) \ne (0,0) \\ 0 & \text{if}(x,y) = (0,0) \end{cases}$$

   Prove that

$$\int_0^1 \int_0^1 f(x,y)\, dy\, dx = \frac{\pi}{4}$$

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = -\frac{\pi}{4}$$

   and

$$\int_0^1 \int_0^1 |f(x,y)|\, dy\, dx = \infty.$$

4. Let X be a r.v. Show that the following conditions are equivalent:
   (i) For each $a \in \mathbb{R}$, $\mathbb{P}\{X = a\} = 0$.
   (ii) For any r.v. Y independent of X, $\mathbb{P}\{X = Y\} = 0$.

5. Let X and Y be two independent r.v.'s. Let $p \ge 1$. Suppose that $E[|X|^p] < \infty$ and $E[|Y|^p] < \infty$. Show that $E[|X + E[Y]|^p] \le E[|X + Y|^p]$.

6. Let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be independent sequences of r.v.'s. Let $0 \le u < t$. Show that

$$\mathbb{P}\{\sup_{n\ge1} |X_n| \ge t\} \inf_{n\ge1} \mathbb{P}\{|Y_n| < u\} \le \mathbb{P}\{\sup_{n\ge1} |X_n + Y_n| \ge t - u\}.$$

7. Suppose X and Y are independent continuous r.v.s with corresponding c.d.f.'s F_X and F_Y. Suppose that for each $a \in \mathbb{R}$, that $F_X(a) \le F_Y(a)$. Show that $P(X \ge Y) \ge 1/2$.

8. Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let f, g be two nonnegative measurable functions such that $f \in L_p$, $g \in L_p$. Prove that

$$\int |fg|\, d\mu = \|f\|_p \|g\|_q,$$

if and only if

$$\|g\|_q^q |f|^p = \|f\|_p^p |g|^q \quad \text{a.e.}$$

9. Let $1 < p < \infty$. Let f, g be two measurable functions such that $f, g \in L_p$. Prove that $\|f + g\|_p = \|f\|_p + \|g\|_p$ if and only if there are nonnegative constants s, t, not both zero, such that $sf = tg$ a.e.

10. Let X be a r.v. in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for some $p_0 > 0$, $E[|X|^{p_0}] < \infty$ and $E[\frac{1}{|X|^{p_0}}] < \infty$. Show that $\lim_{p\to0}(E[|X|^p])^{1/p} = e^{E[\log|X|]}$.