Math 501. 4–th Homework. Due Wednesday, October 31, 2007.

Homework on "Basic properties of Random variables and Expectation".

1. A real estate agent is selling a house. If the house is sold within the first month after the home hits the market, the real estate agent makes \$3000. If the house is sold within the second month, the real estate agent makes \$2000. If the house is sold within the third month, the real estate agent breaks even. If the house is not sold by three months, the real estate agent losses \$4000. The probability that the house is sold within the i–th month is $\frac{1}{2^i}$, for $i = 1, 2, \dots$. What is the real estate agent expected profit?

2. Find two r.v.'s X_1 and X_2 such that:

   (i) For each $a < b$, $\mathbb{P}[a < X_1 \le b, a < X_2 \le b] = \mathbb{P}[a < X_1 \le b]\mathbb{P}[a < X_2 \le b]$.

   (ii) X_1 and X_2 are not independent r.v.'s.

3. Find two r.v.'s X_1 and X_2 such that:

   (i) Both X_1 and X_2 have a continuous distribution.

   (ii) $X_1 + X_2$ does not have a continuous distribution.

4. The random variables X and Y have joint density function

$$f(x,y) = \begin{cases} 3x & \text{if } 0 \le x \le y \le 2x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

   (i) Find $P[Y \ge 1]$.
   (ii) Find $P[Y \ge 1 \ge 3X]$.

5. A **median** m of a r.v. X is a value such that $\mathbb{P}[X < m] \le \frac{1}{2} \le \mathbb{P}[X \le m]$. Let F be the cdf of a r.v. X. Let $m_1 = \sup\{t \in \mathbb{R} : F(t) < \frac{1}{2}\}$ and let $m_2 = \inf\{t \in \mathbb{R} : F(t) > \frac{1}{2}\}$. Show that:
   (a) $-\infty < m_1 \le m_2 < \infty$.
   (b) m is a median of X if and only if $m_1 \le m \le m_2$.

6. Consider a r.v. X with finite first moment. Let $\phi(a) = E[|X - a|]$, $a \in \mathbb{R}$. Show that:

   (i) For each $a \in \mathbb{R}$, $\phi(a) < \infty$.

   (ii) ϕ is a continuous function.

   (iii) $\lim_{|a| \to \infty} \phi(a) = \infty$ is a continuous function.

   (iv) $\phi(a) = \inf_{x \in \mathbb{R}} \phi(x)$ if and only if a is a median of X.

7. Let X be a r.v. with cdf

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x+1}{8} & \text{if } 0 \le x < 2 \\ \frac{(x+1)^2+10}{40} & \text{if } 2 \le x < 4 \\ 1 & \text{if } 4 \le x \end{cases}$$

   Find the mean, the median, and the variance of X.

8. Let Y_1 and Y_2 be two jointly continuous random variables with joint density function

$$f(y_1, y_2) = \begin{cases} 24y_1y_2 & \text{if } 0 \leq y_1, 0 \leq y_2, y_1 + y_2 \leq 1, \\ 0 & else. \end{cases}$$

(i) Find the mean and the variance of Y_1 and Y_2.
(ii) Find the covariance and the correlation coefficient of Y_1 and Y_2.

9. Let X and Y be two random variables satisfying that

$$\mathrm{Var}(X) = 2, \mathrm{Var}(Y) = 5 \text{ and } \mathrm{Var}(-3 + 2X - 3Y) = 17.$$

Find the covariance of X and Y.

10. Let X and Y be two independent random variables with moment generating functions given by

$$M_X(t) = e^{t+t^2} \text{ and } M_Y(t) = e^{2t+2t^2}.$$

Find the mean and the variance of the random variable $Z = 2X - Y$.

11. Suppose that X_1 and X_2 are two independent r.v.'s. Show that if X_1 and $X_1 + X_2$ have the same distribution, then $\mathbb{P}\{X_1 = 0\} = 1$.

12. Suppose that X_1, X_2 and X_3 are three independent identically distributed r.v.'s. Show that if $X_1 + X_2$ and $X_1 + X_3$ are independent, then X_1 has a degenerate distribution.