Math 501. 6–th Homework. Due Friday, November 30, 2007.

Homework on "Common discrete and continuous distributions".

Name: ....

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that there exists $R > 0$ such that $\sum_{n=0}^{\infty} |a_n| R^n < \infty$. Then,

   (i) For each $|x| \le R$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ exist.

   (ii) For each $|x| < R$, $g(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$ exist.

   (iii) For each $|x| < R$, $f'(x) = g(x)$.

   (iv) For each $n \ge 0$, $a_n = \frac{f^{(n)}(0)}{n!}$

2. Let $0 < p \le 1$. Let Y be a r.v. Suppose that:

   (i) $\mathbb{P}[Y \in \mathbb{N}] = 1$.

   (ii) For each $k, n \in \mathbb{N}$,

   $$\mathbb{P}[Y \ge k + n \mid Y \ge k + 1] = \mathbb{P}[Y \ge n].$$

   Show that for each $k \ge 1$, $\mathbb{P}[Y = k] = (1 - p)^{k-1} p$, where $p = \mathbb{P}[Y = 1]$.

3. Let X_n be a binomial distribution with parameters n and p_n. Suppose that $\lim_{n\to\infty} n p_n = \lambda$, where $\lambda > 0$. Let Y be a Poisson r.v. with mean $\lambda > 0$. Show that for each integer $k \ge 0$,

   $$\lim_{n\to\infty} \mathbb{P}\{X_n = k\} = \mathbb{P}\{Y = k\}.$$

4. Let X and Y be two independent r.v.'s with a Poisson distribution and respective parameters λ_1 and λ_2. Show that the conditional distribution of X given $X + Y = n$ is binomial with parameters n and $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.

5. Let X and Y be two independent r.v.s taking values in the nonnegative integers such that

   $$P(X = k | X + Y = n) = \binom{n}{k} p^k (1 - p)^{n-k}$$

   for some $0 < p < 1$ and all $0 \le k \le n$. Suppose also that $X + Y$ has a Poisson distribution with parameter $\lambda > 0$. Show that X and Y have a Poisson distributions. Hint: find the ch.f. of X and Y. The joint ch.f. of X and Y is $\varphi_{X,Y}(s, t) = E[e^{isX + itY}]$

6. Let X be a negative binomial r.v. with parameters r and p. Show that the ch.f. of X is $\varphi(t) = \left(\frac{e^{it}}{1-(1-p)e^{it}}\right)^r$, $t \in \mathbb{R}$.
7. Let X be a positive r.v. such that for each $a, b > 0$, $\Pr\{X > a + b\} = \Pr\{X > a\} \Pr\{X > b\}$. Show that for each $a \geq 0$, $\mathbb{P}\{X > a\} = (P\{X > 1\})^a$. Conclude that X has an exponential distribution with mean $\frac{1}{-\log P\{X>1\}}$. Hint: first prove that for $a = \frac{k}{n}$, where k, n are positive integers, $\mathbb{P}\{X > \frac{k}{n}\} = (P\{X > 1\})^{\frac{k}{n}}$.
8. Prove that for each integer $n \geq 1$,

$$\int \frac{x^n}{n!} e^{-x}\, dx = -\sum_{j=0}^{n} e^{-x} \frac{x^j}{j!} + c.$$

9. Let X be a Gamma r.v. with parameters α and θ. Show that

$$E[X^r] = \frac{\theta^r \Gamma(r+\alpha)}{\Gamma(\alpha)}.$$

10. Suppose that the distribution of Y, conditional on $X = x$, is $N(x, x^2)$ and that the marginal distribution of X is uniform $(0, 1)$. Find $E[Y]$, $\mathrm{Var}(Y)$ and $\mathrm{Cov}(X, Y)$.
11. Let X, Y be i.i.d.r.v.'s. Suppose that:
(i) for each $s, t \in \mathbb{R}$, $sX + tY$ has the same distribution as $\sqrt{s^2 + t^2} X$;
(ii) $\sigma^2 = \mathrm{Var}(X) > 0$.
Show that X has a normal distribution with mean zero.
Hint: Prove that the ch.f. of X is $\exp(-\frac{\sigma^2 t^2}{2})$.
12. Let X be r.v. with values in $\mathbb{R}^d$ and with multinormal distribution with mean μ and covariance matrix Σ. Let $t_1, t_2 \in \mathbb{R}^d$. Show that $t_1'X$ and $t_2'X$ are independent iff $\mathrm{Cov}(t_1'X, t_2'X) = t_1'\Sigma t_2 = 0$.