Math 501. 7–th Homework. Due Monday, December 10, 2007.

Homework on "Transformations of random variables".

1. Let X be a r.v. uniformly distributed on $(0,1)$. Find a function $g(x)$ such that $Y = g(X)$ has a binomial distribution with parameters $n = 3$ and $p = 1/4$.
2. Let X be a random variable with density function

$$f_X(x) = \begin{cases} \frac{3x^2}{2}, & -1 < x < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

   Find the density of $Y = \sqrt{5 - X^3}$.
3. Find the density of the r.v. $Y = 1 - X^2$, if the density of X is given by

$$f_X(x) = \frac{3}{8}(x+1)^2, \quad -1 < x < 1.$$

4. Let X and Y have joint pdf $f_{X,Y}(x,y) = e^{-y}$; $0 < x < y < \infty$ and zero otherwise. Find the joint pdf of $U = X + Y$ and $V = X$. Find the marginal density of U.
5. Let X and Y have joint density function

$$f(x,y) = \frac{1}{x^2y^2}, x \geq 1, y \geq 1.$$

   Compute the joint density function of $U = XY$ and $V = X/Y$. Find the marginal densities of U and V.
6. If X and Y are independent identically distributed uniform random variables on $(0,1)$, compute the density of $U = X/(X+Y)$.
7. Let X, Y, Z be i.i.d. random variables with uniform distribution on the interval $(0,1)$. Find the density of $\frac{XY}{Z}$.
8. Prove that for each $0 < p < 1$, $0 \leq x < n$, where x, n integers,

$$\Pr\{\text{Bin}(n,p) \leq x\} = \Pr\{\text{Beta}(x+1, n-x) \geq p\}.$$

   Hint: Show that as a function of p, the two sides have the same derivative. The density of a beta distribution with parameters α and β is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 \leq x \leq 1.$$

9. Prove that for each $\theta > 0$, and each positive integer n,

$$\Pr\{\text{Poisson}(\theta) \le n\} = \Pr\{\chi^2(2n+2) \ge 2\theta\}.$$

Hint: Use that

$$\int \frac{x^n}{n!} e^{-x}\, dx = -\sum_{j=0}^{n} e^{-x} \frac{x^j}{j!} + c.$$

10. Suppose that a system has 2 parts. The system functions works only if all the parts work. Let X_i is the lifetime of the i–th part of the system. Suppose that X_i has an exponential distribution with mean θ_i. Find the probability that the first part breaks before the second part breaks does.

11. Suppose that a system has n parts. The system functions works only if all n parts work. Let X_i is the lifetime of the i–th part of the system. Suppose that $X_1, \ldots, X_n$ be independent r.v.'s and that X_i has an exponential distribution with mean θ_i. Let $Y = \min(X_1, \ldots, X_n)$ be the lifetime of the system. Show that Y has an exponential distribution. Find the mean of Y.

12. Let X be a r..v with a beta distribution with parameters $\alpha > 0$ and $\beta > 0$. Find $E[X^a(1-X)^b]$, where $a, b > 0$.

13. Let $X_1, \ldots, X_4, Y_1, \ldots, Y_{10}$ be independent r.v.'s. Suppose that $X_1, \ldots, X_4$ are i.i.d.r.v.'s with a normal distribution with mean 1 and variance 1. Suppose that $Y_1, \ldots, Y_{10}$ are i.i.d.r.v.'s with a normal distribution with mean zero and variance $\frac{1}{4}$. Find the distribution of

$$\sum_{i=1}^{4} (X_i - 1)^2 + 4 \sum_{j=1}^{10} Y_j^2.$$

14. Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s from the density

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{else} \end{cases}$$

where $\theta > 0$.

(i) Find the density of the j–th order statistic $X_{(j)}$.

(ii) Find the mean and the variance of $X_{(j)}$.

15. Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s from the density

$$f(x) = \begin{cases} e^{-x} & \text{if } x \ge 0 \\ 0 & \text{else} \end{cases}$$

(i) Find the density of the j–th order statistic $X_{(j)}$.

(ii) Find the mean and the variance of $X_{(j)}$.