Math–502. 1–th Homework. Due Friday, February 15, 2008.

1. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of r.v.'s defined in the same probability space. Show that there exists a sequence of positive numbers $\{a_n\}$ such that $\frac{X_n}{a_n} \overset{\text{a.s.}}{\to} 0$.

2. Find a sequence $\{X_n\}_{n=1}^{\infty}$ of r.v.'s such that $E[X_n] \to 0$, $\text{Var}(X_n) \to 0$ and $\{X_n\}_{n=1}^{\infty}$ does not converges a.s.

3. Let $\{X_n\}$ be a sequence of i.i.d.r.v.'s with a $N(0,1)$ distribution. Show that $\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1$ a.s. and $\liminf_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = -1$ a.s.

4. Let $\{a_n\}$ be a sequence of positive numbers. Let $\{X_n\}$ be a sequence of i.i.d.r.v.'s with an exponential distribution with mean one. Show that $\liminf_{n\to\infty} \dfrac{X_n}{a_n} = 0$ a.s. if and only if $\sum_{n=1}^{\infty} a_n = \infty$.

5. Find a sequence of r.v.'s $\{(X_n, Y_n)\}$ such that:
   (i) $X_n \overset{d}{\to} X$, for some r.v. X
   (ii) $Y_n \overset{d}{\to} Y$, for some r.v. Y
   (iii) (X_n, Y_n) does not converge in distribution.

6. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of r.v.'s. Let $a \in \mathbb{R}$. Suppose that $n^{1/2}(X_n - a) \overset{d}{\to} Z$, where Z is a some r.v. Show that for $n^{1/2}(e^{X_n} - e^a) \overset{d}{\to} e^a Z$.

7. Let $\{X_n)\}$ be a sequence of r.v.'s suppose that $X_n \overset{d}{\to} X$, for some r.v. X with a continuous distribution function. Show that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |P\{X_n \le x\} - \mathrm{P}\{X \le x\}| = 0.$$

8. Prove or give a counterexample to: if $\{X_n\}_{n=1}^{\infty}$ is a sequence of integer valued r.v.'s such that $X_n \overset{d}{\to} X$, for some r.v. X, then $\sum_{k=-\infty}^{\infty} |P\{X_n = k\} - \mathrm{P}\{X = k\}| \to 0$.

9. Prove or give a counterexample to: if $\{X_n\}_{n=1}^{\infty}$ is a sequence of absolutely continuous r.v.'s and $X_n \overset{d}{\to} X$, where X is an absolutely continuous r.v., then $\int_{-\infty}^{\infty} |f_n(x) - f(x)|\,dx \to 0$, where f_n is the p.d.f. of X_n and f is the p.d.f. of X.

10. Find a sequence $\{X_n\}_{n=1}^{\infty}$ of r.v.'s with a normal distribution with mean μ_n and standard deviation σ_n such that $\{X_n\}_{n=1}^{\infty}$ converges in distribution but $\{(\mu_n, \sigma_n)\}_{n=1}^{\infty}$ does not converge.