Math 502. 3–rd Homework. Due Friday, March 7, 2008.

Name: ....

1. Given a random sample of size n from the density

$$f(x|\theta) = \begin{cases} \frac{3x^2}{\theta^3} & \text{if } 0 < x < \theta, \\ 0 & \text{else.} \end{cases}$$

   where $\theta > 0$ is unknown parameter.
   (a) Find the mle of θ.
   (b) Find the method of moments estimator of θ.
   (c) Which of the two estimators above has smallest mean square error?

2. Let $X_1, \dots, X_n$ denote a random sample from the density

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1, \\ 0 & \text{else.} \end{cases}$$

   where $\theta > 0$.
   (a) Find the mle $\hat{\theta}_n$ of θ.
   (b) Prove that this estimator is a consistent estimator of θ.
   (c) Show that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\text{d}} N(0, \sigma^2(\theta))$, for some function $\sigma^2(\theta)$. Determine $\sigma^2(\theta)$.

3. For the distribution in 1.
   (a) Find the method of moments estimator $\hat{\theta}_n^*$ of θ.
   (b) Prove that this estimator is a consistent estimator of θ.
   (c) Show that $\sqrt{n}(\hat{\theta}_n^* - \theta) \xrightarrow{\text{d}} N(0, (\sigma^*(\theta))^2)$, for some function $\sigma^{2,*}(\theta)$. Determine $(\sigma^*(\theta))^2$.
   (d) Which one is smaller, $\sigma^2(\theta)$ or $(\sigma^*(\theta))^2$, where $\sigma^2(\theta)$ is defined in Problem 1(c)?

4. Let $X_1, \dots, X_n$ be a random sample from a distribution with finite fourth moment.
   (a) Show that $\frac{1}{n}\sum_{j=1}^n (X_j - \bar{X})^4$ is a consistent estimator of $E[(X-\mu)^4]$.
   (b) Find $E[\frac{1}{n}\sum_{j=1}^n (X_j - \bar{X})^4]$. Show that $\frac{1}{n}\sum_{j=1}^n (X_j - \bar{X})^4$ is a biased estimator of $E[(X-\mu)^4]$.

5. Let $X_1, \dots, X_n$ be a random sample from a distribution with finite fourth moment. Suppose that the population mean μ is known.
   (a) Find the MSE of $\frac{1}{n-1}\sum_{j=1}^n (X_j - \bar{X})^2$ as an estimator of σ^2.
   (b) Show that $\sqrt{n}(\frac{1}{n-1}\sum_{j=1}^n (X_j - \bar{X})^2 - \sigma^2) \xrightarrow{\text{d}} N(0, S)$. Determine S.
   (Notice that the distribution is not necessarily normal.)

6. Given a random sample of size n from the density

$$f(x) = \begin{cases} \frac{4\theta^3}{x^4} & \text{if } \theta < x, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter.
(a) Find the method of moments estimator $\hat{\theta}_n$ of θ.
(b) Prove that this estimator consistent of θ.
(c) Show that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\text{d}} N(0, \sigma^2(\theta))$, for some function $\sigma^2(\theta)$. Determine $\sigma^2(\theta)$.

7. Given a random sample of size n from the density

$$f(x) = \begin{cases} \frac{4\theta^3}{x^4} & \text{if } \theta < x, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter.
(a) Find the mle $\hat{\theta}_n$ of θ.
(b) Prove that this estimator is a biased estimator of θ.
(c) Find a constant a such that $aX_{(1)}$ is an unbiased estimator of θ.
(d) Find the constant b_0 such that $b_0 X_{(1)}$ minimizes $MSE(bX_{(1)}, \theta)$, over $b \in I\!R$.

8. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution. Show that the MSE of $\frac{1}{n}\sum_{j=1}^{n}(X_j - \mu)^2$ as an estimator of σ^2 is smaller than the MSE of $s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{x})^2$ as an estimator of σ^2.

9. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance $\sigma^2 > 0$. Find the constant a_0 such that $a_0 \frac{1}{n}\sum_{j=1}^{n}(X_j - \bar{X})^2$ minimizes $MSE(a\frac{1}{n}\sum_{j=1}^{n}(X_j - \bar{X})^2, \sigma^2)$, over $a \in I\!R$.

10. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance $\sigma^2 > 0$. Find a such that $as = a\sqrt{\frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2}$ is an unbiased estimator of σ.