Math 502. 4–th Homework. Due Monday, March 17, 2008.

Name: ....

1. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of r.v.'s with mean one and variance four. Show that

$$\sqrt{n}\left(\frac{\sum_{j=1}^{n} X_j + 1}{n-3+\sum_{j=1}^{n} X_j} - \frac{n+1}{2n-5}\right) \xrightarrow{\mathrm{d}} N(0,\sigma^2),$$

for some σ^2. Determine σ^2.

2. Let $X_1,\dots,X_n$ denote a random sample from the density

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$.
(i) Find the Fisher information number.
(ii) Find the Cramér–Rao lower bound for the unbiased estimators of θ.
(iii) Find $\hat{\theta}$ the MLE of θ. Find its variance. Show that $\hat{\theta}$ does not attain the Cramér–Rao lower bound for the unbiased estimators of θ.

3. Let $X_1,\dots,X_n$ be a random sample from a geometric distribution with parameter $0 < p < 1$, i.e. $P\{X = x\} = (1-p)^x p$, for $x = 0,1,\dots,$ Find the mle of p. Find the Cramer–Rao lower bound for the unbiased estimators of p. Is the CRLB attained for the unbiased estimators of p?

4. Let $X_1,\dots,X_n$ be a random sample from $f(x|\theta) = \theta(1+x)^{-(1+\theta)}$, $x,\theta > 0$.
(i) Find the Cramer–Rao lower bound for the unbiased estimators of θ. Is the CRLB for the unbiased estimators of θ attained?
(ii) Find the Cramer–Rao lower bound for the unbiased estimators of θ. Is the CRLB for the unbiased estimators of θ^2 attained?

5. Let $X_1,\dots,X_n$ be a random sample from the pdf $f(x|\theta) = \frac{1}{2\theta}e^{-\frac{|x|}{\theta}}$, $x \in \mathbb{R}, \theta > 0$.
(i) Find the Cramer–Rao lower bound for the unbiased estimators of θ. Is the CRLB for the unbiased estimators of θ attained?
(ii) Find the Cramer–Rao lower bound for the unbiased estimators of θ. Is the CRLB for the unbiased estimators of θ^2 attained?

6. Let $\{X_n\}$ be a sequence of i.i.d.r.v.'s with finite second moment and positive variance. Let $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$ and let $s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{x})^2$. Show that

$$n^{1/2}\left(\frac{\bar{X}}{s} - \frac{\mu}{\sigma}\right) \xrightarrow{\mathrm{d}} N(0, b^2)$$

for some b^2. Find b^2.

7. Let $\{X_n\}$ be a sequence of i.i.d.r.v.'s with finite sixth moment. The sample the skewness of X is defined as $\hat{k}_3 = \frac{\frac{1}{n}\sum_{j=1}^{n}(X_j - \bar{X})^3}{(\frac{1}{n}\sum_{j=1}^{n}(X_j - \bar{X})^2)^{3/2}}$. The population skewness is defined as $k_3 = \frac{E[(X-\mu)^3]}{\sigma^3}$. Find the limit distribution of

$$n^{1/2}(\hat{k}_3 - k_3).$$

8. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{1}{\theta} m x^{m-1} e^{-x^m/\theta}$, $x > 0$, where $m > 0$ is known and $\theta > 0$ is unknown. Find the mle $\hat{\theta}$ of θ. Find the CRLB for the unbiased estimators of θ. Show that $\hat{\theta}$ is an unbiased estimator of θ. Show that the mle attains the CRLB. Show that for each $\theta > 0$,

$$n^{1/2}(\hat{\theta} - \theta) \xrightarrow{\mathrm{d}} N\left(0, \frac{1}{I(\theta)}\right).$$

9. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{1}{\theta} m x^{m-1} e^{-x^m/\theta}$, $x > 0$, where $m > 0$ is known and $\theta > 0$ is unknown.
(a) Find the method of moments estimator $\hat{\theta}_n$ of θ. Show that $n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{\mathrm{d}} N(0, b_1(\theta))$, for some $b_1(\theta)$. Determine $b_1(\theta)$.
(b) Find the maximum likelihood estimator $\hat{\theta}_n$ of θ. Show that $n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{\mathrm{d}} N(0, b_2(\theta))$, for some $b_2(\theta)$. Determine $b_2(\theta)$.
(c) For which values of θ, $b_2(\theta) < b_1(\theta)$?

10. Given a random sample of size n from the density

$$f(x) = \begin{cases} \frac{3\theta^3}{x^4} & \text{if } \theta < x, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter.
(a) Find the method of moments estimator $\hat{\theta}_n$ of θ. Show that $n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{\mathrm{d}} N(0, b(\theta))$, for some $b(\theta)$. Determine $b(\theta)$.
(b) Find the maximum likelihood estimator $\hat{\theta}_n$ of θ. Show that $n(\hat{\theta}_n - \theta) \xrightarrow{\mathrm{d}} U$, for some r.v. U. Determine the cdf of U.

11. Let $X_1, \ldots, X_n$ denote a random sample from the density

$$f(x|\theta) = \begin{cases} \frac{\alpha x^{\alpha-1}}{\theta^\alpha} & \text{if } 0 < x < \theta, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown and $\alpha > 0$ is a known parameter.

(a) Find the method of moments estimator $\hat{\theta}_n$ of θ. Show that $n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{\text{d}} N(0, b(\theta))$, for some $b(\theta)$. Determine $b(\theta)$.

(b) Find the maximum likelihood estimator $\hat{\theta}_n$ of θ. Show that $n(\hat{\theta}_n - \theta) \xrightarrow{\text{d}} U$, for some r.v. U. Determine the cdf of U.