Math 502. 5–th Homework. Due Friday, April 4, 2008.

Show all your work. If a question says find a complete sufficient statistic, you need to prove that the statistic is complete and sufficient.

1. Let $X_1, \dots, X_n$ be a random sample from a Poisson distribution with parameter, $\theta > 0$. Find a minimal sufficient statistic for θ. Which one of the next statistics are sufficient for θ: $T_1(X_1, \dots, X_n) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$, $T_2(X_1, \dots, X_n) = \sum_{i=1}^n X_i$, $T_3(X_1, \dots, X_n) = \sum_{i=1}^n X_i^2$? Why?

2. Let $X_1, \dots, X_n$ be a random sample from a beta distribution with parameters α and β: $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$
$$f(x|\alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$
Find a minimal sufficient statistic for (α, β). Find a complete sufficient statistic for (α, β).

3. Let $X_1, \dots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ with μ and σ^2 unknown. Find the UMVUE of σ^p, where $p > 0$.

4. Let $X_1, \dots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ with μ and σ^2 unknown. Find the UMVUE of σ^4. Find the variance of the UMVUE of σ^4. Find the CRLB for the unbiased estimators of σ^4. Compare the variance of the UMVUE of σ^4 with the CRLB for the unbiased estimators of σ^4.

5. Let $X_1, \dots, X_n$ be a random sample from a $U[\theta, \theta+1]$. Show that $T(X_1, \dots, X_n) = (\min_{1 \le i \le n} X_i, \max_{1 \le i \le n} X_i)$ is a minimal sufficient statistic but it is not complete.

6. Let $X_1, \dots, X_n$ be a random sample from $f(x|\theta) = \frac{2x}{\theta^2}$, $0 < x < \theta$. Find a complete sufficient statistic for θ. Justify that the statistic is complete and sufficient. Find the UMVUE of θ. Find the UMVUE of θ^2.

7. Suppose that X_1 and X_2 are iid observations from the pdf $f(x, \alpha) = \alpha x^{\alpha-1} e^{-x^\alpha}$, $x > 0$, $\alpha > 0$. Show that $(\log X_1)/(\log X_2)$ is an ancillary statistic.

8. Consider a random sample of size n from a Gamma$(2, \beta)$, and let $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ and $\tilde{X} = (\prod_{i=1}^n X_i)^{1/n}$. Show that $T = \bar{X}/\tilde{X}$ is an ancillary statistic for β.

9. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on $(0, \theta)$. Show that $X_{(n)}$ is a complete and sufficient statistic for θ. Find the UMVUE for θ^p, where $p > 0$.

10. Let $X_1, \ldots, X_n$ be a random sample from a exponential distribution with parameter $\theta > 0$. Find the CRLB for the unbiased estimators of θ^2. Find the UMVUE for θ^2. Show that the variance of UMVUE for θ^2 is bigger than the CRLB for the unbiased estimators of θ^2.

11. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{2\theta^2}{x^3}$, if $x > \theta > 0$. Find a complete sufficient statistic. Find a UMVUE for θ. Find a UMVUE for θ^2. Show that $T(X_1, \ldots, X_n) = \frac{\bar{x}}{s}$ is an ancillary statistic, where $\bar{x} = \frac{1}{n}\sum_{j=1}^{n} X_j$ and $s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{x})^2$. Show that $X_{(1)}$ and $T(X_1, \ldots, X_n)$ are independent

12. Let $X_1, \ldots, X_n$ be a random sample from the density $f(x|\theta) = e^{-(x-\theta)}$, $x > \theta$. Find a complete and sufficient statistic for θ. Find a UMVUE for θ. Show that $X_{(1)} = \min_{1 \leq i \leq} X_i$ is a complete sufficient statistic. Show that $X_{(1)}$ and s^2 are independent. Let $X_{(n)} = \max(X_1, \ldots, X_n)$ and let $R = X_{(n)} - X_{(1)}$. R is called the sample range. Show that $X_{(1)}$ and R are independent.