Math 502. 6–th Homework. Due Friday, April 18, 2008.

1. In a random sample of 300 persons eating lunch at a department store cafeteria, only 102 had dessert. Find a 95 % level confidence interval for the average number of people which take dessert at lunch using the following methods:

   (i)

   $$\hat{p} - z_{\alpha/2}\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}} \le p \le \hat{p} + z_{\alpha/2}\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}},$$

   where $\hat{p} = \bar{X}$.

   (ii)

   $$\frac{T}{T+(n+1-T)F(2n+2-2T,2T,\alpha_2)} \le p \le \frac{(T+1)F(2T+2,2n-2T,\alpha/2)}{n-T+(T+1)F(2T+2,2n-2T,\alpha/2)},$$

   where $T = \sum_{j=1}^{n} X_n$.

   (iii) Compare the previous confidence intervals.

2. Let X and let Y equal the concentration in parts per billion of chromium in the blood for healthy persons and for persons with a suspected disease, respectively. Assume that the distribution of X and Y are $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$ respectively. Using $n = 9$ observations of X

   15 23 12 18 9 28 22 11 10

   and $m = 10$ observations of Y

   25 20 35 15 40 16 10 22 18 32

   (a) Give a point estimate of σ_X^2/σ_Y^2.
   (b) Find a 95 % confidence interval for σ_X^2/σ_Y^2.

3. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d.r.v.'s. Let μ be the population mean. An aymptotic level $1-\alpha$ confidence interval for μ is given by

   $$\bar{x} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}.$$

   where $\bar{x}$ is the sample mean and s is the sample stardard deviation. Do and run Splus programs to find 10,000 95 % level confidence intervals for the μ, using the a random sample of size 10 from each of the following distributions: normal, exponential with mean 1, and uniform with minimum 0 and maximum 1. Do a program for each of the distributions. Find the proportion of the confidence intervals which contain the true mean. Are these proportions bigger, smaller or approximately .95?

4. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d.r.v.'s from a normal distribution with variance $\sigma^2 = 9$. A level $1-\alpha$ confidence interval for μ is given by

$$\bar{x} - z_{\frac{\alpha}{2}}\frac{3}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\frac{\alpha}{2}}\frac{3}{\sqrt{n}}.$$

An alternative level $1-\alpha$ confidence interval for μ is given by

$$\bar{x} - t_{n-1,\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{n-1,\frac{\alpha}{2}}\frac{s}{\sqrt{n}},$$

where $\bar{x}$ is the sample mean and s is the sample standard deviation. Do and run a Splus program to find 5,000 95 % level confidence intervals for the μ, using the a random sample of size 20 using the two different confidence intervals. Use the simulations to estimate the coverage probabilities using the two different methods and to estimate the length of the confidence intervals. Compare the coverage probabilities and the length of the confidence intervals form the two methods. What do you conclude?

5. Let $X_1, \ldots, X_n$ be a random sample froma $N(\mu, \sigma^2)$. If σ^2 is known, find a minimum value for n to guarantee that a .95 confidence interval for μ will have length no more than $\sigma/4$.

6. A hospital administrator wishes to estimate the mean number of days that infants spend in ICUs.
(a) How many records should she examine to have 99% confidence that the estimate is not more the 0.5 days from the mean? Previous work suggest that the standard deviation is 1.6.
(b) How many records should she examine if she wants to lower the confidence interval to 95%?

7. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{1}{2\theta}e^{-\frac{|x|}{\theta}}$, $x \in I\!R, \theta > 0$. Find a complete sufficient statistic T for θ. Find the shortest length confidence interval based on a pivotal quantity based on T.

8. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{1}{\theta x^{1+\theta}}$, $x > 1$, $\theta > 0$. Find a complete sufficient statistic T for θ. Find the shortest length confidence interval based on a pivotal quantity based on T.

9. Given a random sample of size n from the density

$$f(x|\theta) = \begin{cases} \frac{m x^{m-1} e^{-\frac{x^m}{\theta}}}{\theta} & \text{if } 0 < x, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter and $m > 0$ is known. Show that the maximum likelihood estimator of θ is $\hat{\theta} = \frac{\sum_{j=1}^{n} X_j^m}{n}$. Show that an approximate large sample $(1-\alpha)\%$ confidence interval for θ is $\hat{\theta} \pm z_{\alpha/2}\frac{\hat{\theta}}{\sqrt{n}}$.

10. Let $X_1, \ldots, X_m$ denote a random sample of size n from a Poisson distribution with mean λ. Find an asymptotic $100(1-\alpha)$ % confidence interval for $\tau(\lambda) = e^{-\lambda} = P(X = 0)$ based on a complete sufficient statistic.