Math 553. 1st Homework. Due Tuesday, February 6, 2001.

Homework on Chapter 1, Introduction to Nonparametrics.

Name: ....

Show all your work. Please, send me the Splus programs which you do my e-mail.

1. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d.r.v.'s. Let μ be the population mean. Assuming normal distribution, the usual $1-\alpha$ confidence interval for μ is given by

$$\bar{x} - t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}.$$

where $\bar{x}$ is the sample mean, s is the sample stardard deviation, $t_{1-\frac{\alpha}{2}}(n-1)$ is the upper $1-\frac{\alpha}{2}$ quantile of a t distribution with $n-1$ degrees of freedom. Show that if $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d.r.v.'s with finite second moment, then

$$\Pr\{\bar{x} - t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}\} \to 1-\alpha.$$

2. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d.r.v.'s. Let σ^2 be the population variance. Assuming normal distribution, the usual $1-\alpha$ confidence interval for σ^2 is given by

$$\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}(n-1)} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}.$$

where s^2 is the sample variance, s is the sample standard deviation, $\chi^2_p(n-1)$ is the upper p quantile of a χ^2 distribution with $n-1$ degrees of freedom. Show that if $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d.r.v.'s with finite fourth moment, then

$$\Pr\left\{\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}(n-1)} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}\right\} \to \Pr\{-z_{\frac{\alpha}{2}} \leq \sqrt{\frac{\kappa-1}{2}}N(0,1) \leq z_{\frac{\alpha}{2}}\}.$$

where $\kappa_4 = \frac{E[(X-\mu)^4]}{\sigma^4}$ is the kurtosis. Show that if $\kappa_4 > 3$, then

$$\Pr\{-z_{\frac{\alpha}{2}} \leq \sqrt{\frac{\kappa_4-1}{2}}N(0,1) \leq z_{\frac{\alpha}{2}}\} < 1-\alpha,$$

and if $\kappa_4 < 3$, then

$$\Pr\{-z_{\frac{\alpha}{2}} \leq \sqrt{\frac{\kappa_4-1}{2}}N(0,1) \leq z_{\frac{\alpha}{2}}\} > 1-\alpha.$$

3. (a) Find the kurtosis for a gamma r.v. with parameters α and β, i.e.

$$f(x, \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}, x \geq 0.$$

(b) Find the kurtosis for a r.v. uniformly distributed on the interval (a, b), where $a < b$.

4. Do and run Splus programs to find 10,000 95 % level confidence intervals for the sample variance using the formula

$$\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}(n-1)} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}$$

using the a random sample of size 10 from each of the following distributions: normal, gamma distribution with mean=2 and variance=2, and uniform with minimum=0 and maximum =1. Do a program for each of the distributions. Find the proportion of the confidence intervals which contain the true variance. Are these proportions bigger, smaller or approximately .95? Comments on your results. Do they agree with questions 2 and 3?