Math 553. 2nd Homework. Due Thursday, February 15, 2001.

Homework on Chapter 2, Inference from a Binomial Distribution.

Name: ....

Show all your work. Please, send me the Splus programs which you do my e-mail.

1. Let S_n be a r.v. with a binomial distribution with parameters n and p. Show that $\frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow{w} N(0,1)$, as $n \to \infty$, using charateristic functions.

2. Do a Splus program to find the value of n so that for the test $H_0 : p = 0.3$ versus $H_a : p = .4$, we have $\alpha = 0.05$ and power $1 - \beta = 0.8$ Using the following methods:
(a) the minimum value of n giving $\alpha < 0.05$ and power > 0.8.
(b) $n = \left(\frac{z_\alpha \sqrt{p_0(1-p_0)} + z_\beta \sqrt{p_1(1-p_1)}}{p_1 - p_0} \right)^2$.
(c) $n = \frac{b + \sqrt{b^2 + 4a}}{4a^2}$, where $b = z_\alpha \sqrt{p_0(1-p_0)} + z_\beta \sqrt{p_1(1-p_1)}$ and $a = p_1 - p_0$.
(d) $n = \left(\frac{z_\alpha \sqrt{p_0(1-p_0)} + z_\beta \sqrt{p_1(1-p_1)}}{p_1 - p_0} \right)^2 + \frac{2}{p_1 - p_0}$.
Which of the last three methods approaches better (a)?

3. Prove that for each $0 < p < 1$, $0 \leq x < n$, where x, n integers,

$$\Pr\{\text{Bin}(n, p) \leq x\} = \Pr\{\text{Beta}(x + 1, n - x) \geq p\}.$$

Hint: Show that as a function of p, the two sides have the same derivative. The density of a beta distribution with parameters α and β is

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha - 1}(1 - x)^{\beta - 1}, 0 \leq x \leq 1.$$

4. Let X be a r.v. with a F distribution with degrees of freedom p, q. Show that $Y = \frac{\frac{p}{q} F_{p,q}}{1 + \frac{p}{q} F_{p,q}}$ has a beta distribution with parameters $\frac{p}{2}$ and $\frac{q}{2}$. Hint: if h is a one-to-one function, X is a r.v. and $Y = h(X)$, then the density of y is given by $f_y(y) = f_X(h^{-1}(y))|\frac{dh^{-1}(y)}{dy}|$, where $f_X(x)$ is the density of X. Recall that the density of an F distribution with degrees of freedom p, q is

$$f_{p,q}(x) = \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \left(\frac{p}{q} \right)^{p/2} \frac{x^{(p-2)/2}}{\left(1 + \frac{p}{q} x\right)^{\frac{p+q}{2}}}, 0 \leq x.$$

5. Let $X = x$ have a binomial distribution with parameters n and p. Do and run Splus programs to find 10,000 95 % level confidence intervals for p: (a) Using the normal

approximation $\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ (b) using the Clopper–Pearson method. Do a program for each of the methods. Find the proportion of the confidence intervals which contain the true variance for each of the methods. Are these proportions bigger, smaller or approximately .95? Comments on your results. Which method is more accurate?