Math–571. 1st Homework. Due Thursday, February 8, 2007.

1. Let $(\Omega, \mathcal{F})$ be a measurable space. A finitely additive function μ on $\mathcal{F}$ is a function $\mu : \mathcal{F} \to I\!R$ such that $\mu[\cup_{j=1}^m A_j] = \sum_{j=1}^m \mu[A_j]$, for each $\{A_j\}_{j=1}^m \subset \mathcal{F}$, which are pairwise disjoint. Let μ be a nonnegative finitely additive function on $\mathcal{F}$ such that if $\{A_n\}$ is a sequence in $\mathcal{F}$ such that $A_n \searrow \emptyset$, then $\mu(A_n) \searrow 0$. Show that μ is a measure in $(\Omega, \mathcal{F})$.

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let

$$\overline{\mathcal{F}} = \{A \cup B \in \Omega : A \in \mathcal{F}, \text{ there exists } C \in \mathcal{F}, \text{ such that } B \subset C \text{ and } \mu(C) = 0\}.$$

(i) Show that $\overline{\mathcal{F}}$ is a σ–field.

(ii) Given $A \cup B \in \overline{\mathcal{F}}$, define $\overline{\mu}(A \cup B) = \mu(A)$, where $A \in \mathcal{F}$ and there exists $C \in \mathcal{F}$, such that $B \subset C$ and $\mu(C) = 0$. Show that $\overline{\mu}$ is well defined.

(iii) Show that $\overline{\mu}$ is a complete measure on $\overline{\mathcal{F}}$.

(iv) Show that for each $A \in \mathcal{F}$, $\overline{\mu}(A) = \mu(A)$.

3. Show that for any events $A_1, \ldots, A_n$,

$$\mathrm{P}[\cap_{j=1}^n A_j] \ge \sum_{j=1}^n \mathrm{P}[A_j] - (n-1).$$

4. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability measurable space. Let $\{A_n\}$ be a sequence of measurable events such that $\mathrm{P}[A_n] = 1$ for each $n \ge 1$. Show that $\mathrm{P}[\cap_{n=1}^\infty A_n] = 1$.

5. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let $A_1, \ldots, A_n \in \mathcal{F}$, where $n \ge 2$. Prove that

$$\mathrm{P}[\cup_{i=1}^n A_i] \ge \sum_{i=1}^n \mathrm{P}[A_i] - \sum_{1 \le i_1 < i_2 \le n} \mathrm{P}[A_{i_1} \cap A_{i_2}].$$