Math–571. 2-nd Homework. Due Tuesday February 20, 2007.

1. Given $A \subset \mathbb{R}$, let

$$m^*(A) = \inf\{\sum_{j=1}^{\infty}(b_j - a_j) : A \subset \cup_{j=1}^{\infty}(a_j, b_j), a_j < b_j, a_j, b_j \in \mathbb{R}\}.$$

   Show that:
   (i) m^* is an outer measure on $\mathbb{R}$.
   (ii) $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}(\mu^*)$.
   (iii) For each $a < b$, $m^*[a, b] = b - a$.

2. Given a measure μ on $(\Omega, \mathcal{F})$, the outer measure μ^* of μ is defined as

$$\mu^*(E) = \inf\{\mu(A) : E \subset A, A \in \mathcal{F}\}, E \subset \Omega.$$

   Show that for each set $E \subset \Omega$, there exists a set $A \in \mathcal{F}$ such that $E \subset A$ and $\mu^*(E) = \mu(A)$.

3. Let m be the Lebesgue measure in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Let A be a Borel set of $\mathbb{R}^d$ and let $a \in \mathbb{R}^d$. Prove that $a + A$ is a Borel set of $\mathbb{R}^d$ and $m(A) = m(a + A)$.

4. Suppose that a r.v. X satisfies that for each $a \in \mathbb{R}$, $\mathbb{P}\{X \le a\} = \mathbb{P}\{-X \le a\}$. Show that for each set $A \in \mathcal{B}$, $\mathbb{P}\{X \in A\} = \mathbb{P}\{-X \in A\}$.

5. Let $X_1, \ldots, X_n$ be independent identically distributed r.v.'s. Show that for each set $A \in \mathcal{B}(\mathbb{R}^n)$ and each permutation σ of $\{1, \ldots, n\}$, $\mathbb{P}\{(X_1, \ldots, X_n) \in A\} = \mathbb{P}\{(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) \in A\}$.

6. Prove or give a counterexample of the following proposition: " Let X_1 and X_2 be two r.v.'s. Show that X_1 and X_2 and independent r.v.'s if and only if for each $a < b$, $\mathbb{P}[a < X_1 \le b, a < X_2 \le b] = \mathbb{P}[a < X_1 \le b]\mathbb{P}[a < X_2 \le b]$."