Math–571. 3–th Homework. Due Tuesday, March 6, 2007.

1. Define $f:[0,1]\to I\!R$ by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x^2} & \text{if } 0<x\le 1,\\ 0 & \text{if } x=0.\end{cases}$$

Show that f is continuous in $[0,1]$ and differentiable in $(0,1)$. Show that $\int_0^1 f'(x)\,dx$ does not exists.

2. Let $f:[0,1]\to I\!R$ be the function defined by

$$f(x)=\begin{cases} 0 & \text{if } x \text{ is rational},\\ 1 & \text{else}.\end{cases}$$

Show that f is Borel measurable. Find $\int_0^1 f(x)\,dx$.

3. (i) A differentiable function φ on an open interval I is convex on I if and only if φ' is nondecreasing on I. (ii) A second differentiable function φ on an open interval I is convex on I if and only if $\varphi''\ge 0$ on I.

4. Let X be a r.v. Let $I=\{p\ge 0: E[|X|^p]<\infty\}$.
(i) Given $a>0$, find a r.v. such that $\{p\ge 0: E[|X|^p]<\infty\}=[0,a)$.
(ii) Given $a>0$, find a r.v. such that $\{p\ge 0: E[|X|^p]<\infty\}=[0,a]$.
(iii) Find a r.v. such that $\{p\ge 0: E[|X|^p]<\infty\}=[0,\infty)$.

5. Let $\varphi: I\to I\!R$ be a function, where I is an open interval of $I\!R$. Suppose that for each $a\in I$, there exists a constant β_a such that $\beta_a(x-a)\le\varphi(x)-\varphi(a)$ for each $x\in I$. Show that φ is convex. Prove that a function $\varphi: I\to I\!R$ is convex iff there exists a set T and $a_t, b_t\in I\!R$ such that for each $x\in I$, $\varphi(x)=\sup_{t\in T}(a_t+b_t x)$.

6. Let X be a r.v. in a probability space $(\Omega,\mathcal{F},\mathrm{P})$. Prove that if $X\in L_\infty(\Omega,\mathcal{F},\mathrm{P})$, then $\lim_{p\to\infty}\|X\|_p=\|X\|_\infty$ and $\lim_{n\to\infty}\frac{E[|X|^{p+1}]}{E[|X|^p]}=\|X\|_\infty$. Prove that if $X\notin L_\infty(\Omega,\mathcal{F},\mathrm{P})$, then $\lim_{p\to\infty}\|X\|_p=\infty$.

7. (Extended dominated convergence theorem) Let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be two sequences of measurable functions on the measure space $(\Omega,\mathcal{F},\mu)$. Suppose that:
(i) $|f_n|\le g_n$ a.e. (μ).
(ii) $f_n\to f$ a.e.

(iii) $g_n \to g$ a.e.
(iv) $\int g_n \, d\mu \to \int g \, d\mu < \infty$.
Show that $\int f_n \, d\mu \to \int f \, d\mu$.

8. Let X and Y be two r.v.'s. Let $p > 0$. Show that:

   (i) If $E[|X|^p] < \infty$ and $E[|X|^p] < \infty$, then $E[|X+Y|^p] < \infty$.

   (ii) If X and Y are independent and $E[|X+Y|^p] < \infty$, then $E[|X|^p] < \infty$ and $E[|X|^p] < \infty$.

   (iii) Find two r.v.'s X and Y such that $E[|X+Y|^p] < \infty$ but $E[|X|^p] < \infty$.

9. Let f be the function in $[0,1] \times [0,1]$ defined by

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } 2^{-n} < x \leq 2^{-n+1}, 2^{-n} < y \leq 2^{-n+1} \text{ for some } n \geq 1 \\ -2^{2n+1} & \text{if } 2^{-n-1} < x \leq 2^{-n}, 2^{-n} < y \leq 2^{-n+1} \text{ for some } n \geq 1 \\ 0 & \text{else} \end{cases}$$

   Show that

$$\int_0^1 \int_0^1 f(x,y) \, dx \, dy = 0, \int_0^1 \int_0^1 f(x,y) \, dy \, dx = 1$$

   and

$$\int_0^1 \int_0^1 |f(x,y)| \, dy \, dx = \infty.$$

10. Find the mean and the variance of the r.v. with c.d.f.

$$F(x) = \begin{cases} 0 & \text{if } \; x < 0, \\ \frac{x+2}{8} & \text{if } 0 \leq x < 1, \\ \frac{3x^2+4}{16} & \text{if } 1 \leq x < 2, \\ 1 & \text{if } \; 2 \leq x. \end{cases}$$

   Warning: X has a mixed distribution.