Math–571. 4–th Homework. Due Tuesday, March 20, 2007.

1. Let $L_0(\Omega,\mathcal{F},P)$ be the vector space consisting by the r.v.'s defined on the probability space $(\Omega,\mathcal{F},P)$. Let $\mathcal{L}_0(\Omega,\mathcal{F},P)$ be the classes of equivalence of $L_0(\Omega,\mathcal{F},P)$ with respect to the equivalence relation $X \sim Y$ iff $X = Y$ a.s. Show that there is no norm $\|\cdot\|$ on $\mathcal{L}_0(\Omega,\mathcal{F},P)$ such that $X_n \stackrel{P}{\to} X$ iff $\|X_n - X\| \to 0$.

2. Prove that there exists no distance d in $\mathcal{L}_0(\Omega,\mathcal{F},\mu)$ such that $d(X_n,X) \to 0$ iff $X_n \stackrel{\text{a.s.}}{\to} X$.

3. Consider the probability space $([0,1],\mathcal{B}([0,1]),m)$, where m is the Lebesgue measure. Let $X_n = \frac{n}{\log n} I(0,\frac{1}{n})$, $n \ge 3$. Show that $\{X_n\}_{n=1}^\infty$ is uniformly integrable, $X_n \stackrel{\text{Pr}}{\to} 0$ and $E[X_n] \to 0$, but $E[\sup_{n\ge 1}|X_n|] = \infty$.

4. Consider the probability space $([0,1],\mathcal{B}([0,1]),m)$, where m is the Lebesgue measure. Let $X_n = nI(0,\frac{1}{n}) - nI(\frac{1}{n},\frac{2}{n})$. Show that $X_n \stackrel{\text{Pr}}{\to} 0$ and $E[X_n] \to 0$, but $\{X_n\}_{n=1}^\infty$ is not uniformly integrable,

5. Let $\{X_n\}_{n=1}^\infty$ be a uniformly integrable collection of r.v.'s. Show that $\left\{\frac{\sum_{j=1}^n X_j}{n}\right\}_{n=1}^\infty$ is uniformly integrable.

6. Let $\{X_n\}_{n=1}^\infty$ be a sequence of r.v.'s defined in the same probability space. Show that there exists a sequence of positive numbers $\{a_n\}_{n=1}^\infty$ such that $\frac{X_n}{a_n} \stackrel{\text{a.s.}}{\to} 0$.

7. Let $\{X_n\}_{n=1}^\infty$ be a sequence of nonnegative r.v.'s. Suppose that that for some $0 < \alpha < \beta$, $E[X_n^\alpha] \to 1$ and $E[X_n^\beta] \to 1$. Show that $X_n \stackrel{\text{Pr}}{\to} 1$.

8. Prove or give a counterexample to: if $\{F_n\}_{n=1}^\infty$ is a sequence of c.d.f.'s such that for each $x \in \mathbb{R}$, $\lim_{m,n\to\infty}(F_n(x) - F_m(x)) = 0$, then there exists a c.d.f. F such that $F_n \stackrel{d}{\to} F$.

9. Prove or give a counterexample to: if $\{X_n\}_{n=1}^\infty$ is a sequence of absolutely continuous r.v.'s and $X_n \stackrel{d}{\to} X$, where X is an absolutely continuous r.v., then $\int_{-\infty}^\infty |f_n(x) - f(x)|\,dx \to 0$, where f_n is the p.d.f. of X_n and f is the p.d.f. of X.

10. Prove or give a counterexample to: if $\{X_n\}_{n=1}^\infty$ is a sequence of integer valued r.v.'s such that $X_n \stackrel{d}{\to} X$, for some r.v. X, then $\sum_{k=-\infty}^\infty |P\{X_n = k\} - \mathrm{P}\{X = k\}| \to 0$.