Math–571. 5–th Homework. Due Tuesday, April 10, 2007.

1. Let $\{X_n\}$ be a sequence of r.v.'s. Let $S_n = \sum_{j=1}^n X_j$. Let $p \ge 1$. Show that:
(i) $X_n \overset{\text{a.s.}}{\to} 0$ implies $\frac{S_n}{n} \overset{\text{a.s.}}{\to} 0$.
(ii) $X_n \overset{L_p}{\to} 0$ implies $\frac{S_n}{n} \overset{L_p}{\to} 0$.
(iii) $X_n \overset{\text{P}}{\to} 0$ does not imply $\frac{S_n}{n} \overset{\text{P}}{\to} 0$.

2. Let $\{X_j\}_{j=1}^\infty$ be a sequence of nonnegative independent r.v.'s. Prove that the following conditions are equivalent:
(i) $\sum_{j=1}^\infty X_j$ converges a.s.
(ii) $\sum_{j=1}^\infty \text{P}\{X_j \ge 1\} < \infty$ and $\sum_{j=1}^\infty E[X_j I(X_j < 1)] < \infty$.
(iii) $\sum_{j=1}^\infty E\left[\frac{X_j}{X_j+1}\right] < \infty$.

3. Let X be a r.v. such that for each $A \in \mathcal{B}(I\!R)$, $\text{P}\{X \in A\}$ is either zero or one. Show that there exists $c \in I\!R$ such that $P\{X = c\} = 1$.

4. Let $\{X_n\}_{n=1}^\infty$ be a sequence of i.i.d.r.v.'s from a distribution with mean zero. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers. Show that $\sum_{n=1}^\infty a_n X_n$ converges a.s. if and only if $\sum_{n=1}^\infty a_n^2 < \infty$.

5. Let $\{g_n\}_{n=1}^\infty$ be a sequence of i.i.d.r.v.'s from a standard normal distribution. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers. Show that $\sum_{j=1}^\infty a_n g_n^2$ converges a.s. if and only if $\lim_{n\to\infty} \sum_{j=1}^n a_j$ exists and $\sum_{n=1}^\infty a_n^2 < \infty$.

6. Let X_1 and X_2 be two independent r.v.'s. Suppose that $X_1 + X_2$ is a degenerate r.v. Show that X_1 and X_2 are degenerate.

7. Let $\{X_n\}$ be a sequence of i.i.d.r.v.'s. Prove that $\limsup_{n\to\infty} \frac{|X_n|}{n} = \infty$ a.s. if and only if $E[|X_1|] = \infty$.

8. Let $\{X_n\}$ be a sequence of i.i.d.r.v.'s. Show that $\limsup_{n\to\infty} |X_n| = \|X_1\|_\infty$ a.s.

9. Show that if $X \overset{\text{d}}{\sim} N(0,1)$, then for each $x > 0$,

$$\frac{x}{1+x^2}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \le \text{P}\{X \ge x\} \le e^{-\frac{x^2}{2}}.$$

Hint: use that that for any r.v. X and any $t > 0$,

$$\text{P}\{X \ge t\} \le \inf_{\lambda \ge 0} e^{-\lambda t} E[e^{\lambda X}].$$

10. Let $\{X_n\}$ be a sequence of i.i.d.r.v.'s with a $N(0,1)$ distribution. Show that $\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1$ a.s. and $\liminf_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = -1$ a.s.