Math–571. 6–th Homework. Due Thursday, April 26, 2007.

1. Let $\varphi(t)$ be the ch.f. of a r.v. X. Show that if $|\varphi(t)| = 1$ for each t in a neighborhood of zero, then X is degenerate.
2. Show that $\varphi(t) = \frac{1}{1+t^2}$ is a ch.f.
3. Prove that for each $\alpha > 2$, $\varphi(t) = e^{-|t|^\alpha}$ is not a ch.f.
4. Let $\{X_{n,j} : 1 \le j \le k_n, n \ge 1\}$ be a triangular array of is rowwise independent r.v.'s. Suppose that the distribution of $X_{n,j}$ is uniform in the interval $(-j, j+n)$. Find constants a_n, b_n such that $\frac{S_n - b_n}{a_n} \xrightarrow{d} N(0,1)$. You need to prove that $\frac{S_n - b_n}{a_n} \xrightarrow{d} N(0,1)$.
5. Let $\{X_j\}$ be a sequence of independent r.v.'s. Suppose that:
   (i) There exists a constant M such that $|X_j| \le M$ a.s. for each $j \ge 1$.
   (ii) $\sum_{j=1}^\infty \mathrm{Var}(X_j) = \infty$.
   Then, $\frac{\sum_{j=1}^n (X_j - E[X_j])}{\sqrt{\sum_{j=1}^n \mathrm{Var}(X_j)}} \xrightarrow{\mathrm{d}} N(0,1)$.
6. Let $\{X_j\}$ be a sequence of independent r.v.'s. Suppose that:
   (i) For each $j \ge 1$, $E[X_j] = 0$ and $E[X_j^2] = 1$.
   (ii)There are constants M and $p > 2$ such that $E[|X_j|^p] \le M$, for each $j \ge 1$.
   Then, $n^{-1/2} \sum_{j=1}^n (X_j - E[X_j]) \xrightarrow{\mathrm{d}} N(0,1)$.
7. Let $\{X_j\}$ be a sequence of independent r.v.'s such that
   $$\mathrm{P}\{X_j = -j\} = \mathrm{P}\{X_j = j\} = \frac{1}{2j^2}, \text{ and } \mathrm{P}\{X_j = -1\} = \mathrm{P}\{X_j = 1\} = \frac{j^2 - 1}{2j^2}.$$
   Show that $n^{-1} \sum_{j=1}^n \mathrm{Var}(X_j) \to 2$, $n^{-1/2} \sum_{j=1}^n (X_j - E[X_j]) \xrightarrow{\mathrm{d}} N(0,1)$ and the conditions in the Lindeberg–Feller theorem are not satisfied.
8. Let $\{X_j\}$ be a sequence of independent r.v.'s such that
   $$\mathrm{P}\{X_j = -j^{1/2}\} = \mathrm{P}\{X_j = j^{1/2}\} = \frac{1}{2j}, \text{ and } \mathrm{P}\{X_j = 0\} = \frac{j-1}{j}.$$
   Show that $E[X_j] = 0$, $\mathrm{Var}(X_j) = 1$ and $n^{-1/2} \sum_{j=1}^n (X_j - E[X_j])$ converges in distribution, but the limit does not have a normal distribution.
9. Let $\{X_j\}$ be a sequence of symmetric i.i.d.r.v.'s with
   $$\Pr\{|X_1| \ge t\} = \frac{1}{(t+1)\ln((t+1)e)} \text{ for } t \ge 0.$$
   Show that $n^{-1} \sum_{j=1}^n X_j \xrightarrow{\mathrm{P}} 0$ but, it is not true that $n^{-1} \sum_{j=1}^n X_j \xrightarrow{\text{a.s.}} 0$.
10. Let $\{X_j\}$ be a sequence of symmetric i.i.d.r.v.'s with finite fourth moment. Find the limit distribution of $\{(U_n, V_n)'\}$, where $U_n := n^{-1/2} \sum_{j=1}^n (X_j - \mu)$, $V_n := n^{-1/2} \sum_{j=1}^n ((X_j - \bar{X})^2 - \sigma)$, $\bar{X} := \frac{1}{n} \sum_{j=1}^n X_j$, $\mu := E[X_1]$ and $\sigma^2 := \mathrm{Var}(X_1)$.