Math–571. 7–th Homework. Due Tuesday, May 15, 2007.

1. Find a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, a r.v. X on Ω and a sub–σ–field $\mathcal{A}$ of $\mathcal{F}$ such that $E[X|\mathcal{A}] = 0$ a.s., but it is not true that $X = 0$ a.s.

2. Suppose that X_1, X_2 and X_3 are independent r.v.'s. Show that $E[X_1X_3|\sigma(X_1, X_2)] = X_1E[X_3]$.

3. Let X be an absolutely continuous r.v. with pdf f_X. Show that

$$E[X||X| = x] = \frac{f_X(x) - f_X(-x)}{f_X(x) + f_X(-x)}|x|, \text{ if } x > 0 \text{ and } f_X(x) + f_X(-x) > 0,$$

$$E[X||X| = 0] = 0$$

4. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let $\mathcal{G}_i$, for $i = 1, 2, 3$ be three σ–fields of $\mathcal{F}$. Show that the following conditions are equivalent:

   (i) $\mathrm{P}(A|\sigma(\mathcal{G}_1, \mathcal{G}_2)) = \mathrm{P}(A|\mathcal{G}_2)$, for each $A \in \mathcal{G}_3$.

   (ii) $\mathrm{P}(B \cap C|\mathcal{G}_2) = \mathrm{P}(B|\mathcal{G}_2)\mathrm{P}(C|\mathcal{G}_2)$, for each $B \in \mathcal{G}_1$ and each $C \in \mathcal{G}_3$.

5. Let $\{(X_n, \mathcal{F}_n)\}_{n=1}^\infty$ be a martingale. Prove that $\{(X_n, \mathcal{G}_n)\}_{n=1}^\infty$ is a martingale, where $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$.

6. $\{(X_n, \mathcal{F}_n)\}_{n=1}^\infty$ be a submartingale. Let τ_1 and τ_2 be two finite stopping times. Suppose that

   (i) $\tau_1 \le \tau_2$.

   (ii) $E[\tau_2] < \infty$, $E[|X_{\tau_1}|] < \infty$ and $E[|X_{\tau_2}|] < \infty$.

   (iii) $\liminf_{n\to\infty} E[X_n^+ I(\tau_2 > n)] = 0$.

   Then, $E[X_{\tau_2}|\mathcal{F}_{\tau_1}] \ge X_{\tau_1}$ a.s.

7. Let $\{X_n\}_{n=1}^\infty$ be a sequence of i.id.r.v.'s with $\mathrm{P}\{X_n = 1\} = \mathrm{P}\{X_n = -1\} = \frac{1}{2}$. Let $\tau := \inf\{n \ge 1 : S_n = 1\}$. Show that $E[\tau] = \infty$.

8. Suppose that $\{X_n\}$ is a martingale in L_2. Show that $E[(X_n - E[X_n])^2] = \sum_{j=1}^n E[(X_j - X_{j-1})^2]$, where $X_0 = E[X_1]$.

9. Prove that every nonnegative supermartingale converges almost surely.

10. Let $\{X_j\}_{j=1}^\infty$ be a sequence of independent r.v.'s such that $E[X_j] = 0$ and $E[X_j^2] < \infty$. Show that $\{(S_n^2 - \sigma_n^2, \mathcal{F}_n)\}$ is a martingale, where $S_n = \sum_{j=1}^n X_j$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, and $\sigma_n^2 = \sum_{j=1}^n E[X_j^2]$.