Manual for SOA Exam FM/CAS Exam 2.

Chapter 1. Basic Interest Theory.
Section 1.2. Simple interest.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2,
Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Simple interest

Under simple interest:

- the interest paid over certain period of time is proportional to the length of this period of time and the principal.

Simple interest

Under simple interest:

- the interest paid over certain period of time is proportional to the length of this period of time and the principal.
- if i is the effective annual rate of simple interest, the amount of interest earned by a deposit of k held for t years is kit. The balance at time t years is $k + kit = k(1 + it)$.

Simple interest

Under simple interest:

- the interest paid over certain period of time is proportional to the length of this period of time and the principal.
- if i is the effective annual rate of simple interest, the amount of interest earned by a deposit of k held for t years is kit. The balance at time t years is $k + kit = k(1 + it)$.
- interest is found using the principal not the earned interest. To find the earned interest, we need to know the amount of principal, not the balance.

Simple interest

Under simple interest:

- the interest paid over certain period of time is proportional to the length of this period of time and the principal.
- if i is the effective annual rate of simple interest, the amount of interest earned by a deposit of k held for t years is kit. The balance at time t years is $k + kit = k(1 + it)$.
- interest is found using the principal not the earned interest. To find the earned interest, we need to know the amount of principal, not the balance.
- balances under simple interest follow the proportionality rule and rule about the addition of several deposits/withdrawals. However, the rule "grows–depends–on–balance" does not hold.

Suppose that an account earns simple interest with annual effective rate of i.

- If an investment of 1 is made at time zero, then the balance in this account at time t years is $a(t) = 1 + it$.

Suppose that an account earns simple interest with annual effective rate of i.

- If an investment of 1 is made at time zero, then the balance in this account at time t years is $a(t) = 1 + it$.
- If an investment of k is made at time zero, then the balance in this account at time t years is $k(1 + it)$.

Suppose that an account earns simple interest with annual effective rate of i.

- If an investment of 1 is made at time zero, then the balance in this account at time t years is $a(t) = 1 + it$.
- If an investment of k is made at time zero, then the balance in this account at time t years is $k(1 + it)$.
- If an investment of k is made at time s years, then the balance in this account at time t years, $t > s$, is $k(1 + i(t - s))$. Notice that the investment is held for $t - s$ years, and the earned interest is $ki(t - s)$.

Notice that the amount $k(1+i(t-s))$ is not $\frac{ka(t)}{a(s)} = \frac{k(1+it)}{(1+is)}$. Making an investment of $\frac{k}{(1+is)}$ at time zero, we have a balance of $\frac{k}{(1+is)}(1+is) = k$ at time s. Making an investment of $\frac{k}{(1+is)}$ at time zero, we have a balance of $\frac{k}{(1+is)}(1+it)$ at time t. This is not the balance at time t years in an account with an investment of k made at time s years.

- Making an investment of $\frac{k}{(1+is)}$ at time zero, we have a balance of $\frac{k}{(1+is)}(1+is) = k$ at time s. But since interest does not earn interest, the amount of interest earned in the period $[s, t]$ is $\frac{k}{(1+is)}i(t-s)$. Hence, the balance at time t is

$$k + \frac{k}{(1+is)}i(t-s) = \frac{k(1+is) + ki(t-s)}{(1+is)} = \frac{k(1+it)}{(1+is)}.$$

- Making an investment of k at time s years, we have a balance of $k(1+i(t-s))$ at time t.

Present value for simple interest

- A deposit of k made at time s has a future value of $k(1+i(t-s))$ at time t, if $t>s$.

Present value for simple interest

- A deposit of k made at time s has a future value of $k(1 + i(t - s))$ at time t, if $t > s$.
- To get a balance of k time s, we need to make a deposit of $k\frac{1}{1+i(s-t)}$ at time t, if $t < s$.

Present value for simple interest

Theorem 1

If deposits/withdrawals are make according with the table,

Deposits	C_1	C_2	$\cdots$	C_n
Time	t_1	t_2	$\cdots$	t_n

where $0 \leq t_1 < t_2 < \cdots < t_n$ *to an account earning simple interest with annual effective rate of* i*, then the balance at time* t *years, where* $t > t_n$*, is given by*

$$B = \sum_{j=1}^{n} C_j(1 + i(t - t_j)) = \sum_{j=1}^{n} C_j + \sum_{j=1}^{n} C_j i(t - t_j).$$

Proof.

Time	Deposit/withdr. at that time	Principal after the deposit	Amount of interest earned up to that time
t_1	C_1	C_1	0
t_2	C_2	$C_1 + C_2$	$C_1 i(t_2 - t_1)$
t_3	C_3	$\sum_{j=1}^{3} C_j$	$\sum_{j=1}^{2} C_j i(t_3 - t_j)$
$\dots$	$\dots$	$\dots$	$\dots$
t_k	C_k	$\sum_{j=1}^{k} C_j$	$\sum_{j=1}^{k-1} C_j i(t_k - t_j)$
$\dots$	$\dots$	$\dots$	$\dots$
t_n	C_n	$\sum_{j=1}^{n} C_j$	$\sum_{j=1}^{n-1} C_j i(t_n - t_j)$
t	0	$\sum_{j=1}^{n} C_j$	$\sum_{j=1}^{n} C_j i(t - t_j)$

The amount of interest earned up to time t_3 is

$$C_1 i(t_2 - t_1) + (C_1 + C_2) i(t_3 - t_2) = C_1 i(t_3 - t_1) + C_2 i(t_3 - t_2)$$
$$= \sum_{j=1}^{2} C_j i(t_3 - t_j).$$

The amount of interest earned up to time t_k is the amount of interest earned up to time t_{k-1} plus the amount of interest earned in the period $[t_{k-1}, t_k]$, which is

$$\sum_{j=1}^{k-2} C_j i(t_{k-1} - t_j) + \sum_{j=1}^{k-1} C_j i(t_k - t_{k-1})$$
$$= \sum_{j=1}^{k-1} C_j i(t_{k-1} - t_j) + \sum_{j=1}^{k-1} C_j i(t_k - t_{k-1})$$
$$= \sum_{j=1}^{k-1} C_j i(t_k - t_j).$$

Theorem 2

If deposits/withdrawals are make according with the table,

Deposits	C_1	C_2	$\cdots$	C_n
Time	t_1	t_2	$\cdots$	t_n

where $0 \le t_1 < t_2 < \cdots < t_n$ *to an account earning simple interest and the balance at time* t *years, where* $t > t_n$*, is* B*, then the annual effective rate of* i

$$i = \frac{B - \sum_{j=1}^{n} C_j}{\sum_{j=1}^{n} C_j(t - t_j)}.$$

Proof. Solving for i in $B = \sum_{j=1}^{n} C_j + \sum_{j=1}^{n} C_j i(t - t_j)$, we get the value of i.

In the formula,

$$i = \frac{B - \sum_{j=1}^{n} C_j}{\sum_{j=1}^{n} C_j(t - t_j)},$$

$B - \sum_{j=1}^{n} C_j$ is the total amount of interest earned, $\sum_{j=1}^{n} C_j(t - t_j)$ is the sum of the balances times the amount balances are in the account.

Example 3

Jeremy invests \$1000 into a bank account which pays simple interest with an annual rate of 7%. Nine months later, Jeremy withdraws \$600 from the account. Find the balance in Jeremy's account one year after the first deposit was made.

Example 3

Jeremy invests \$1000 into a bank account which pays simple interest with an annual rate of 7%. Nine months later, Jeremy withdraws \$600 from the account. Find the balance in Jeremy's account one year after the first deposit was made.

Solution: The cashflow of deposits is

deposit/withdrawal	1000	−600
Time (in years)	0	0.75

.

The balance one year after the first deposit was made is

$$\sum_{j=1}^{n} C_j(1+i(t-t_j))$$
$$=(1000)(1+(1-0)(0.07))+(-600)(1+(1-0.75)(0.07))$$
$$=459.5.$$

Time	Deposit made at this time	Principal after deposit	Amount of interest earned in the last period
0	1000	1000	0
0.75	-600	400	$(1000)(0.07)(0.75) = 52.5$
1	0	400	$(400)(0.07)(1-0.75) = 7$

The balance one year after the first deposit was made is

$$400 + 52.5 + 7 = 459.5.$$

Example 4

On September 1, 2006, John invested $25000 into a bank account which pays simple interest. On March 1, 2007, John's wife made a withdrawal of 5000. The accumulated value of the bank account on July 1, 2007 was $20575. Calculate the annual effective rate of interest earned by this account.

Example 4

On September 1, 2006, John invested \$25000 into a bank account which pays simple interest. On March 1, 2007, John's wife made a withdrawal of 5000. The accumulated value of the bank account on July 1, 2007 was \$20575. Calculate the annual effective rate of interest earned by this account.

Solution: Let September 1, 2006 be time 0. Then, March 1, 2007 is time $\frac{6}{12}$ years; and July 1, 2007 is time $\frac{10}{12}$ years. The annual effective rate of interest earned by this account is

$$i = \frac{B - \sum_{j=1}^{n} C_j}{\sum_{j=1}^{n} C_j(t - t_j)} = \frac{20575 - 25000 + 5000}{25000\left(\frac{10}{12}\right) - 5000\left(\frac{10}{12} - \frac{6}{12}\right)}$$
$$= \frac{575}{19166.66667} = 3\%.$$

