Manual for SOA Exam FM/CAS Exam 2.

Chapter 1. Basic Interest Theory.

Section 1.3. Compounded interest.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2,
Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Compound interest

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Under compound interest, the effective rate of interest over a certain period of time depends only on the length of this period, i.e.

$$\text{for each } 0 \leq s < t, \frac{A(t)-A(s)}{A(s)} = \frac{A(t-s)-A(0)}{A(0)}.$$

Notice that

$$\frac{A(t)-A(s)}{A(s)} = \frac{A(0)(1+i)^t - A(0)(1+i)^s}{A(0)(1+i)^s} = (1+i)^{t-s} - 1.$$

The effective rate of interest earned in the n–th year is

$$i_n = \frac{A(n)-A(n-1)}{A(n-1)} = \frac{A(0)(1+i)^n - A(0)(1+i)^{n-1}}{A(0)(1+i)^{n-1}} = i.$$

Under compound interest, the present value at time t of a deposit of k made at time s is

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If deposits/withdrawals are made according with the table

Deposits	C_1	C_2	$\cdots$	C_n
Time (in years)	t_1	t_2	$\cdots$	t_n

where $0 \le t_1 < t_2 < \cdots < t_n$, into an account earning compound interest with an annual effective rate of interest of i, then the present value at time t of the cashflow is

$$V(t) = \sum_{j=1}^{n} C_j (1+i)^{t-t_j}.$$

In particular, the present value of the considered cashflow at time zero is $\sum_{j=1}^{n} C_j(1+i)^{-t_j}$.

Example 1

A loan with an effective annual interest rate of 5.5% is to be repaid with the following payments:
(i) 1000 at the end of the first year.
(ii) 2000 at the end of the second year.
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Solution: The cashflow of payments to the loan is

Payments	1000	2000	5000
Time	1	2	3

The loaned amount at time zero is the present value at time zero of the cashflow of payments, which is

$$(1000)(1.055)^{-1} + (2000)(1.055)^{-2} + (5000)(1.055)^{-3}$$
$$=947.8672986 + 1796.904831 + 4258.068321 = 7002.840451.$$

The accumulation function for simple interest is $a(t) = 1 + it$, which is a linear function.
The accumulation function for compound interest is $a(t) = (1 + i)^t$, which is an increasing convex function.
We have that
(i) If $0 < t < 1$, then $(1 + i)^t < 1 + it$.
(ii) If $1 < t$, then $1 + it < (1 + i)^t$.

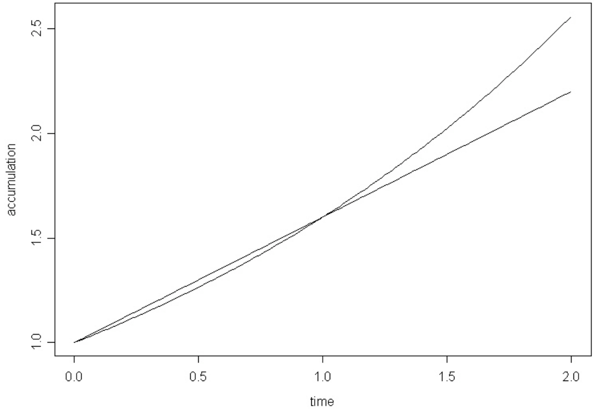

Figure 1: comparison of simple and compound accumulation functions

Usually, we solve for variables in the formula, $A(t) = A(0)(1+i)^t$, using the TI–BA–II-Plus calculator.

To turn on the calculator press $\boxed{\text{ON/OFF}}$.

To clear errors press $\boxed{\text{CE/C}}$. It clears the current displays (including error messages) and tentative operations.

When entering a number, you realized that you make a mistake you can clear the whole display by pressing $\boxed{\text{CE/C}}$.

When entering numbers, if you would like to save some of the entered digits, you can press $\boxed{\rightarrow}$ as many times as digits you would like to remove. Digits are deleted starting from the last entered digit.

It is **recommended** to set–up the TI-BA–II–Plus calculator to 9 decimals. You can do that doing

$\boxed{\text{2nd}}$, $\boxed{\text{FORMAT}}$, $\boxed{9}$, $\boxed{\text{ENTER}}$, $\boxed{\text{2nd}}$, $\boxed{\text{QUIT}}$.

We often will use **the time value of the money worksheet** of the calculator. There are 5 main **financial variables** in this worksheet:

- The number of periods $\boxed{\text{N}}$.
- The nominal interest for year $\boxed{\text{I/Y}}$.
- The present value $\boxed{\text{PV}}$.
- The payment per period $\boxed{\text{PMT}}$.
- The future value $\boxed{\text{FV}}$.

You can use the calculator to find one of these financial variables, by entering the rest of the variables in the memory of the calculator and then pressing $\boxed{\text{CPT}}$ $\boxed{\text{financial key}}$, where financial key is either $\boxed{\text{N}}$, $\boxed{\%\ \text{i}}$, $\boxed{\text{PV}}$, $\boxed{\text{PMT}}$ or $\boxed{\text{FV}}$.

Here, $\boxed{\text{financial key}}$ is either $\boxed{\text{N}}$, $\boxed{\%\ \text{i}}$, $\boxed{\text{PV}}$, $\boxed{\text{PMT}}$ or $\boxed{\text{FV}}$.

- You can recall the entries in the time value of the money worksheet, by pressing $\boxed{\text{RCL}}$ $\boxed{\text{financial key}}$.
- To enter a variable in the entry $\boxed{\text{financial key}}$, type the entry and press $\boxed{\text{financial key}}$. The entry of variables can be done in any order.
- To find the value of any of the five variables (after entering the rest of the variables in the memory) press $\boxed{\text{CPT}}$ $\boxed{\text{financial key}}$.
- When computing a variable, a formula using all five variables and two auxiliary variables is used

To set–up $\boxed{\text{C/Y}}$=1 and $\boxed{\text{P/Y}}$=1, do

$$\boxed{\text{2nd}}, \boxed{\text{P/Y}}, \boxed{1}, \boxed{\text{ENTER}}, \boxed{\downarrow}, \boxed{1}, \boxed{\text{ENTER}}, \boxed{\text{2nd}}, \boxed{\text{QUIT}}.$$

To check that this is so, do

$$\boxed{\text{2nd}}\ \boxed{\text{P/Y}}\ \boxed{\downarrow}\ \boxed{\text{2nd}}\ \boxed{\text{QUIT}}.$$

If $\boxed{\text{PMT}}$ equals zero, $\boxed{\text{C/Y}}$=1 and $\boxed{\text{P/Y}}$=1, you have the formula,

$$\boxed{\text{PV}} + \boxed{\text{FV}} \left(1 + \frac{\boxed{\text{I/Y}}}{100}\right)^{-\boxed{\text{N}}} = 0. \tag{1}$$

You can use this to solve for any element of the four elements in the formula $A(t) = A(0)(1+i)^t$. Unless it is said otherwise, we will **assume that the entries for $\boxed{\textbf{C/Y}}$ and $\boxed{\textbf{P/Y}}$ are both 1 and $\boxed{\textbf{PMT}}$ is 0**.

Example 2

Mary invested \$12000 on January 1, 1995. Assuming composite interest at 5 % per year, find the accumulated value on January 1, 2002.

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Solution: $A(t) = 12000(1 + 0.05)^7 = 16885.21$. You can do this in the calculator by entering:
$\boxed{0}$ $\boxed{\text{PMT}}$ $\boxed{7}$ $\boxed{\text{N}}$ $\boxed{5}$ $\boxed{\text{I/Y}}$ $\boxed{12000}$ $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$.
Note that since the calculator, uses the formula

$$\boxed{\text{PV}} + \boxed{\text{FV}} \left(1 + \frac{\boxed{\text{I/Y}}}{100}\right)^{-\boxed{\text{N}}} = 0.$$

the display in your calculator is negative.

Example 3

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Solution: We solve for i in $275 = 200(1+i)^5$ and get $i = 6.5763\%$. In the calculator, you do

$\boxed{-275}$ $\boxed{\text{FV}}$ $\boxed{5}$ $\boxed{\text{N}}$ $\boxed{200}$ $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$.

Since the calculator, uses the formula (1), either the present value or the future value has to be entered as negative number (and the other one as a positive number). If you enter both the present value and the future value as positive values, you get the error message $\boxed{\text{Error 5}}$. To clear this error message press $\boxed{\text{CE/C}}$.

Example 4

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Solution: We solve for t in $275 = 200(1 + 0.05)^t$ and get that $t = 6.5270$ years. In the calculator, you do

$\boxed{-275}$ $\boxed{\text{FV}}$ $\boxed{5}$ $\boxed{\text{I/Y}}$ $\boxed{200}$ $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{N}}$.

Example 5

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Solution: We solve for t in $3 = (1 + 0.08)^t$ and get $t = 14.2749$ years. In the calculator, you do
$\boxed{-3}$ $\boxed{\text{FV}}$ $\boxed{8}$ $\boxed{\text{I/Y}}$ $\boxed{1}$ $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{N}}$.

Example 6

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Solution: We solve for $A(0)$ in $10000 = A(0)(1+0.05)^{10}$ and get that $A(0) = 6139.13$. In the calculator, you do $\boxed{10000}$ $\boxed{\text{FV}}$ $\boxed{5}$ $\boxed{\text{I/Y}}$ $\boxed{10}$ $\boxed{\text{N}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$.

The calculator has a memory worksheet with values in the memory, which stores ten numbers. These ten numbers are called: $\boxed{\text{M0}}$, $\cdots$, $\boxed{\text{M9}}$.To enter the number in the display into the i–th entry of the memory, press $\boxed{\text{STO}}$ $\boxed{i}$, where i is an integer from 0 to 9. To recall the number in the memory entry i, press $\boxed{\text{RCL}}$ $\boxed{i}$, where i is an integer from 0 to 9. The command $\boxed{\text{STO}}$ $\boxed{+}$ $\boxed{i}$ adds the value in display to the entry i in the memory. You can see all the numbers in the memory by accessing the memory worksheet. To enter this worksheet press $\boxed{\text{2nd}}$ $\boxed{\text{MEM}}$. Use the arrows $\boxed{\uparrow}$, $\boxed{\downarrow}$ to move from entry to another. To entry a new value in one entry, type the number and press $\boxed{\text{ENTER}}$.

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Using the calculator, you do
$\boxed{-1000}$ $\boxed{\text{FV}}$ $\boxed{1}$ $\boxed{\text{N}}$ $\boxed{5.5}$ $\boxed{\text{I/Y}}$ $\boxed{\text{CPT}}\boxed{\text{PV}}$
and get $(1000)(1.055)^{-1} = 947.8672986$. You enter this number in the memory of the calculator doing $\boxed{\text{STO}}$ $\boxed{1}$
Next doing
$\boxed{-2000}$ $\boxed{\text{FV}}$ $\boxed{2}$ $\boxed{\text{N}}$ $\boxed{\text{CPT}}\boxed{\text{PV}}$
you find $(2000)(1.055)^{-2} = 1796.904831$. Notice that you do not have to reenter the percentage interest rate. You enter this number in the memory of the calculator doing $\boxed{\text{STO}}$ $\boxed{2}$
Next doing
$\boxed{-5000}$ $\boxed{\text{FV}}$ $\boxed{3}$ $\boxed{\text{N}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$
you get $(5000)(1.055)^{-3} = 4258.068321$. You enter this number in the memory of the calculator doing $\boxed{\text{STO}}$ $\boxed{3}$.
You can recall and add the three numbers doing
$\boxed{\text{CRCL}}$ $\boxed{1}$ $\boxed{+}$ $\boxed{\text{CRCL}}$ $\boxed{2}$ $\boxed{+}$ $\boxed{\text{CRCL}}$ $\boxed{3}$ $\boxed{=}$
and get 7002.840451.

