Manual for SOA Exam FM/CAS Exam 2.

Chapter 1. Basic Interest Theory.

Section 1.6. Force of interest.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Spring 2009 Edition",
available at http://www.actexmadriver.com/

Force of interest

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To find the force of interest, we may use the accumulation function,

$$\frac{d}{dt} \ln A(t) = \frac{d}{dt} \ln(A(0)a(t)) = \frac{d}{dt} \ln(A(0)) + \frac{d}{dt} \ln(a(t))$$
$$= \frac{d}{dt} \ln(a(t)).$$

Example 1

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Solution: We have that

$$\ln(A(t)) = \ln\left(25\left(1 + \frac{t}{4}\right)^3\right) = \ln 25 + 3\ln\left(1 + \frac{t}{4}\right).$$

The force of interest is

$$\delta_t = \frac{d}{dt}\ln(A(t)) = \frac{d}{dt}\left(\ln 25 + 3\ln\left(1 + \frac{t}{4}\right)\right) = 3\frac{\frac{1}{4}}{1 + \frac{t}{4}} = \frac{3}{4 + t}.$$

From the equation, $\frac{3}{4+t} = \frac{1}{2}$, we get that $t = 2$.

The force of interest is also called the **rate of interest continuously compounded** and the **continuous interest rate**. We have that

$$\delta_t = \lim_{h\to 0} \frac{A(t+h)-A(t)}{A(t)\cdot h} = \lim_{h\to 0} \frac{\text{interest earned over the next } h \text{ years}}{\text{investment at time } t\cdot h}.$$

The nominal annual rate earned in the next $\frac{1}{m}$ years compounded m times a year at time t is

$$\frac{m(a\left(t+\frac{1}{m}\right)-a(t))}{a(t)} = \frac{a\left(t+\frac{1}{m}\right)-a(t)}{a(t)\frac{1}{m}}.$$

We have that

$$\lim_{m\to\infty} \frac{a\left(t+\frac{1}{m}\right)-a(t)}{a(t)\frac{1}{m}} = \delta_t.$$

Under compound interest, $a(t) = (1+i)^t$ and

$$\delta_t = \frac{d}{dt}\ln a(t) = \frac{d}{dt}\ln(1+i)^t = \frac{d}{dt} t \ln(1+i) = \ln(1+i)$$

Under compound interest, the force of interest is a constant δ, such that $\delta = \ln(1+i) = -\ln \nu$.
Under compound interest,

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} d^{(m)} = \delta.$$

In the case of simple interest, $a(t) = 1 + it$ and $\delta_t = \frac{d}{dt}\ln(1+it) = \frac{i}{1+it}$. The force of interest is decreasing with t.

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Theorem 2

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Proof.

Since $\delta_s = \frac{d}{ds} \ln a(s)$ and $a(0) = 1$,

$$\int_0^t \delta_s\, ds = \int_0^t \frac{d}{ds} \ln a(s)\, ds = \ln a(s) \Big|_0^t = \ln a(t).$$

So, $a(t) = e^{\int_0^t \delta_s\, ds}$. $\square$

Example 3

A bank account credits interest using a force of interest $\delta_t = \frac{3t^2}{t^3+2}$. A deposit of 100 is made in the account at time $t = 0$. Find the amount of interest earned by the account from the end of the 4–th year until the end of the 8–th year.

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Solution: First, we find $a(t) = e^{\int_0^t \delta_s\, ds}$.

$$\int_0^t \delta_s\, ds = \int_0^t \frac{3s^2}{s^3+2}\, ds = \ln(s^3+2)\Big|_0^t$$
$$= \ln(t^3+2) - \ln 2 = \ln\left(\frac{t^3+2}{2}\right)$$

and

$$a(t) = e^{\int_0^t \delta_s\, ds} = e^{\ln\left(\frac{t^3+2}{2}\right)} = \frac{t^3+2}{2} = 1 + \frac{t^3}{2}.$$

The amount of interest earned in the considered period is

$$100(a(8) - a(4)) = (100)\left(1 + \frac{8^3}{2} - \left(1 + \frac{4^3}{2}\right)\right) = 22400.$$

