Manual for SOA Exam FM/CAS Exam 2.

Chapter 2. Cashflows.

Section 2.3. Yield rates.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Yield of return

Suppose that the future value at time t of the cashflow:

Investments	V_0	C_1	C_2	$\cdots$	C_n
Time	0	t_1	t_2	$\cdots$	t_n

is FV. Then, the **rate of return** i of the investment satisfies the equation,

$$FV = V_0\nu^{-t} + \sum_{j=1}^{n} C_j\nu^{t_j-t} = V_0(1+i)^t + \sum_{j=1}^{n} C_j(1+i)^{t-t_j}.$$

The rate of return i, $i > -1$, solving this equation is called the **yield rate of return** or **internal rate of return**. This equation can have either no solutions, or one solution, or several solutions. We are interested in values of i with $i > -1$. If $i < -1$, then $(1+i)^n > 0$ is n is even and $(1+i)^n < 0$ is n is odd. Values of i with $i \le -1$ do not make any sense.

Example 1

Suppose that John invest \$3000 in a business. One year later, John sells half of this business to a partner for \$6000. Two years after the beginning, the business is in red and John has to pay \$4000 to close this business. What is the rate of interest John's got in his investment?

Example 1

Suppose that John invest \$3000 in a business. One year later, John sells half of this business to a partner for \$6000. Two years after the beginning, the business is in red and John has to pay \$4000 to close this business. What is the rate of interest John's got in his investment?

Solution: The cashflow is:

Inflow	-3000	6000	-4000
Time	0	1	2

Since John lost money, one expect that i is negative. However, there is no solution. We have to solve $-3000(1+i)^2+6000(1+i)-4000=0$, or $3(1+i)^2-6(1+i)+4=0$. Using the quadratic formula,

$$1+i=\frac{6\pm\sqrt{6^2-4\cdot 4\cdot 3}}{2}=\frac{6\pm\sqrt{-12}}{2}.$$

There is no solution.

Example 2

What is the yield rate on a transaction in which a person makes payments of \$100 immediately and \$100 at the end of two years, in exchange for a payment of \$201 at the end of one year? Find all possible solutions.

Example 2

What is the yield rate on a transaction in which a person makes payments of \$100 immediately and \$100 at the end of two years, in exchange for a payment of \$201 at the end of one year? Find all possible solutions.

Solution: The cashflow is:

Inflow	-100	201	-100
Time	0	1	2

We have to solve $-100 + 201(1+i)^{-1} - 100(1+i)^{-2} = 0$, or $100(1+i)^2 - 201(1+i)^1 + 100 = 0$. Using the quadratic formula,

$$1 + i = \frac{201 \pm \sqrt{201^2 - 4 \cdot 100 \cdot 100}}{200} = \frac{201 \pm \sqrt{201}}{200}.$$

The two solutions are $i = 10.5124922\%$ and $i = -9.512492197\%$.

Since the internal rate of return could either do not exist or have several solutions, it is not a good indication of the performance of general investment strategy. However there exists a unique rate of return i with $i > -1$ if either all outflows happen before all the inflows, or all inflows happen before all the outflows.

Suppose that you an investment strategy consisting of investing (positive) payments of $C_1, \dots, C_m$ at times $t_1 < \cdots < t_m$. At times $s_1 < \cdots < s_n$, we get respective (positive) returns $P_1, \dots, P_n$, where $s_1 > t_m$. The cashflow is

Inflows	$-C_1$	$-C_2$	$\cdots$	$-C_m$	P_1	P_2	$\cdots$	P_n
Time	t_1	t_2	$\cdots$	t_m	s_1	s_2	$\cdots$	s_n

In this case, there exists a unique solution to the equation

$$\sum_{k=1}^{n} P_k(1+i)^{-s_k} - \sum_{j=1}^{m} C_j(1+i)^{-t_j} = 0, i > -1.$$

Besides,

- $\sum_{k=1}^{n} P_k > \sum_{j=1}^{m} C_j$, then $i > 0$.
- $\sum_{k=1}^{n} P_k < \sum_{j=1}^{m} C_j$, then $i < 0$.
- $\sum_{k=1}^{n} P_k = \sum_{j=1}^{m} C_j$, then $i = 0$.

Example 3

As the budgeting officer for Road Kill Motors Inc., you are evaluating the purchase of a new car factory. The cost of the factory is \$4 million today. It will provide inflows of \$1.4 million at the end of each of the first three years. Find the effective rate of interest which this investment will provide your company.

Example 3

As the budgeting officer for Road Kill Motors Inc., you are evaluating the purchase of a new car factory. The cost of the factory is \$4 million today. It will provide inflows of \$1.4 million at the end of each of the first three years. Find the effective rate of interest which this investment will provide your company.

Solution: The cashflow is

Contributions	-4	1.4	1.4	1.4
Time	0	1	2	3

An equation of value for the cashflow is

$$0 = 4 - (1.4)(1+i)^{-1} - (1.4)(1+i)^{-2} - (1.4)(1+i)^{-3}.$$

In the TI–BA–II–Plus calculator, we can find i, by going to $\boxed{\text{CF}}$, and enter $\boxed{\text{CFo}}=-4$, $\boxed{\text{C01}}=1.4$, $\boxed{\text{F01}}=3$, $\boxed{\text{2nd}}$, $\boxed{\text{QUIT}}$. We can move between different entries using the arrows $\boxed{\downarrow}$ and $\boxed{\uparrow}$. Press $\boxed{\text{IRR}}$ $\boxed{\text{CPT}}$ and get $IRR = i = 2.47974\%$.

Example 4

Find the internal rate of return such that a payment of 400 at the present, 200 at the end of one year, and 300 at the end of two years, accumulate to 1000 at the end of 3 years.

Example 4

Find the internal rate of return such that a payment of 400 at the present, 200 at the end of one year, and 300 at the end of two years, accumulate to 1000 at the end of 3 years.

Solution: The cashflow is

Contributions	-400	-200	-300	1000
Time	0	1	2	3

An equation of value for the cashflow is

$$0 = -400(1+i)^3 - 200(1+i)^2 - 300(1+i) + 1000.$$

In the TI–BA–II–Plus calculator, we can find i, by going to $\boxed{\text{CF}}$, and enter $\boxed{\text{CFo}}=-400$, $\boxed{\text{C01}}=-200$, $\boxed{\text{F01}}=1$, $\boxed{\text{C02}}=-300$, $\boxed{\text{F02}}=1$, $\boxed{\text{C03}}=1000$, $\boxed{\text{F03}}=1$, $\boxed{\text{2nd}}$, $\boxed{\text{QUIT}}$. We can move between different entries using the arrows $\boxed{\downarrow}$ and $\boxed{\uparrow}$. Press $\boxed{\text{IRR}}$ $\boxed{\text{CPT}}$ and get $IRR = 5.0709\%$.

Example 5

An investment fund is established at time 0 with a deposit of \$5000. \$6000 is added at the end of 6 months. The fund value, including interest, is \$11500 at the end of 1 year. Find the internal rate of return as a annual nominal rate convertible semiannually.

Example 5

An investment fund is established at time 0 with a deposit of \$5000. \$6000 is added at the end of 6 months. The fund value, including interest, is \$11500 at the end of 1 year. Find the internal rate of return as a annual nominal rate convertible semiannually.

Solution: The cashflow is

Investments	5000	6000	11500
Time (in half years)	0	1	2

An equation of value for the cashflow is

$$0 = (5000) + (6000)\left(1+\frac{i^{(2)}}{2}\right)^{-1} - (11500)\left(1+\frac{i^{(2)}}{2}\right)^{-2}.$$

In the TI–BA–II–Plus calculator, press $\boxed{\text{CF}}$, and enter $\boxed{\text{CFo}}$=5000, $\boxed{\text{C01}}$=6000, $\boxed{\text{F01}}$=1, $\boxed{\text{C02}}$=−11500, $\boxed{\text{F02}}$=1. Press $\boxed{\text{IRR}}$, $\boxed{\text{CPT}}$ and get $IRR = \frac{i^{(2)}}{2} = 3.095064303\%$ and $i^{(2)} = 6.190128606\%$. The six–month effective interest rate is $\frac{i^{(2)}}{2}$.

Example 6

An investment fund is established at time 0 with a deposit of \$5000. \$6000 is added at the end of 6 months. The fund value, including interest, is \$11500 at the end of 1 year. Find the internal rate of return as a annual nominal rate convertible monthly.

Example 6

An investment fund is established at time 0 with a deposit of \$5000. \$6000 is added at the end of 6 months. The fund value, including interest, is \$11500 at the end of 1 year. Find the internal rate of return as a annual nominal rate convertible monthly.

Solution: The cashflow is

Investments	5000	6000	11500
Time (in months)	0	6	12

An equation of value for the cashflow is

$$0 = (5000) + (6000)\left(1 + \frac{i^{(12)}}{12}\right)^{-6} - (11500)\left(1 + \frac{i^{(12)}}{12}\right)^{-12}.$$

In the TI–BA–II–Plus calculator, press $\boxed{\text{CF}}$, and enter $\boxed{\text{CFo}}$=5000, $\boxed{\text{C01}}$=0, $\boxed{\text{F01}}$=5, $\boxed{\text{C02}}$=6000, $\boxed{\text{F02}}$=1, $\boxed{\text{C03}}$=0, $\boxed{\text{F03}}$=5, $\boxed{\text{C04}}$=−11500, $\boxed{\text{F04}}$=1. Press $\boxed{\text{IRR}}$, $\boxed{\text{CPT}}$ and get $IRR = \frac{i^{(12)}}{12} = 0.509314804\%$ and $i^{(12)} = 6.111777648\%$.

