Manual for SOA Exam FM/CAS Exam 2.

Chapter 2. Cashflows.

Section 2.4. Dollar–weighted and time–weighted rates of return.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Dollar–weighted and time–weighted rates of return

If the cashflow

Investments	V_0	C_1	C_2	$\cdots$	C_n
Time	0	t_1	t_2	$\cdots$	t_n

has future value FV at time t, then its equation of value is

$$FV = V_0(1+i)^t + \sum_{j=1}^{n} C_j(1+i)^{t-t_j}.$$

Using the first order Taylor expansion $1+it$ of $(1+i)^t$, the previous equation of value is approximately,

$$FV = V_0(1+it) + \sum_{j=1}^{n} C_j(1+i(t-t_j)). \tag{1}$$

Observe Equation (1) represents the future value of the cashflow when simple interest is used.

The interest rate i which solves

$$FFV = V_0(1+it) + \sum_{j=1}^{n} C_j(1+i(t-t_j)).$$

is called the **dollar weighted rate of return**, which is

$$i = \frac{FV - V_0 - \sum_{j=1}^{n} C_j}{V_0 t + \sum_{j=1}^{n}(t-t_j)C_j}. \tag{2}$$

V_0 can be interpreted as the initial balance in an account. C_j is the deposit at time t_j. FV is the final balance in the account at time t. Hence, $I = FV - V_0 - \sum_{j=1}^{n} C_j$ is the interest earned in the account. $V_0 t + \sum_{j=1}^{n}(t-t_j)C_j$ is the sum of the deposits multiplied by the time which the deposits are in the account.

Example 1

On January 1, 2000, the balance in account is \$25200. On April 1, 2000, \$500 are deposited in this account and on July 1, 2001, a withdraw of \$1000 is made. The balance in the account on October 1, 2001 is \$25900. What is the annual rate of interest in this account according with the dollar–weighted method?

Example 1

On January 1, 2000, the balance in account is $25200. On April 1, 2000, $500 are deposited in this account and on July 1, 2001, a withdraw of $1000 is made. The balance in the account on October 1, 2001 is $25900. What is the annual rate of interest in this account according with the dollar–weighted method?

Solution: The cashflow

Investments	25200	500	−1000
Time in years	0	3/12	18/12

has a FV at time 21/12 of $25900. So, the annual dollar–weighted rate of interest is

$$\frac{25900 - 25200 - 500 + 1000}{(25200)(21/12) + (500)(18/12) - 1000(3/12)}$$
$$= \frac{1200}{44100 + 750 - 250} = \frac{1200}{44600} = 2.6906\%$$

Suppose that we make investments in a fund over time and we know the outstanding balance before each deposit or withdrawal occurs. Let B_0 be the initial balance in the fund. Let B_j be the balance in the fund immediately before time t_j, for $1 \leq j \leq n$. Let W_j be the amount of each deposit or withdrawal at time t_j. $W_j > 0$ for deposits and $W_j < 0$ for withdrawal. In a table, we have:

Time	0	t_1	t_2	$\cdots$	t_{n-1}	t_n
Balance before depos./withdr.	–	B_1	B_2	$\cdots$	B_{n-1}	B_n
Depos./Withdr.	–	W_1	W_2	$\cdots$	W_{n-1}	–
Balance after depos./withdr.	B_0	$B_1 + W_1$	$B_2 + W_2$	$\cdots$	$B_{n-1} + W_{n-1}$	–

The **time–weighted annual rate of return** i is the solution of

$$(1+i)^{t_n} = \frac{B_1}{B_0} \cdot \frac{B_2}{B_1 + W_1} \cdot \frac{B_3}{B_2 + W_2} \cdots \frac{B_n}{B_{n-1} + W_{n-1}}.$$

In the j–th period of time, the balance of the fund has changed from $B_{j-1} + W_{j-1}$ to B_j. So, the interest factor rate in the j–th period of time is $1 + i_j = \frac{B_j}{B_{j-1}+W_{j-1}}$, where i_j is the effective rate of return in the period $[t_{j-1}, t_j]$. Observe that if the investment followed an annual effective rate of interest of i, the interest factor from time t_{j-1} to time t_j would be $(1 + i)^{t_j - t_{j-1}}$. Assuming that $1 + i_j = (1 + i)^{t_j - t_{j-1}}$, we get that

$$(1+i_1)(1+i_2)\cdots(1+i_n) = (1+i)^{t_1}(1+i)^{t_2-t_1}\cdots(1+i)^{t_n-t_{n-1}} = (1+i)^{t_n}.$$

The **time–weighted annual rate of return** i is the solution of

$$(1 + i)^{t_n} = \frac{B_1}{B_0} \cdot \frac{B_2}{B_1 + W_1} \cdot \frac{B_3}{B_2 + W_2} \cdots \frac{B_n}{B_{n-1} + W_{n-1}}.$$

Usually, the account balance does not follow compound interest with a fixed effective rate i. Usually, $1 + i_j$ and $(1 + i)^{t_j - t_{j-1}}$ may be different.

Example 2

For an investment account, you are given:

Date	11/1/04	3/1/05	8/1/05	2/1/06	4/1/06
Account Balance (before deposit or withdrawal)	14,516	14,547	18,351	16,969	18,542
Deposit	–	3,000	–	2500	–
Withdrawal	–	–	2,000	–	–

Calculate the annual effective yield rate by the time weighted method.

Example 2

For an investment account, you are given:

Date	*11/1/04*	*3/1/05*	*8/1/05*	*2/1/06*	*4/1/06*
Account Balance (before deposit or withdrawal)	*14,516*	*14,547*	*18,351*	*16,969*	*18,542*
Deposit	–	*3,000*	–	*2500*	–
Withdrawal	–	–	*2,000*	–	–

Calculate the annual effective yield rate by the time weighted method.

Solution: The annual effective yield rate i by the time weighted method satisfies

$$(1+i)^{17/12} = \frac{B_1}{B_0} \cdot \frac{B_2}{B_1+W_1} \cdot \frac{B_3}{B_2+W_2} \cdot s \frac{B_n}{B_{n-1}+W_{n-1}}$$
$$= \frac{14547}{14516} \frac{18351}{14547+3000} \frac{16969}{18351-2000} \frac{18542}{16969+2500} = 1.035877$$

and $i = 2.5193371\%$.

