Manual for SOA Exam FM/CAS Exam 2.

Chapter 2. Cashflows.

Section 2.6. Continuous payments.

Extract from:

"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Continuous payments

Suppose that the payments are made very often. Then by approximation, instead of a sum, we have an integral. It is like the payments are made continuously. Let $V(t)$ be the outstanding fund balance at time t of the cashflow. Assume that contributions are made continuously at an instantaneous rate $C(t)$, then the equation of value is

$$V(t) = V(0)(1+i)^t + \int_0^t C(s)(1+i)^{t-s}\, ds. \tag{1}$$

(1) appears as the limit of the equation of value for the cashflow:

Inflow	$V(0)$	$C(t_1)(t_1-0)$	$C(t_2)(t_2-t_1)$	$\cdots$	$C(t_n)(t_n-t_{n-1})$
Time	0	t_1	t_2	$\cdots$	t_n

as $\max_{1\le j\le n}(t_j-t_{j-1})\to 0$, where
$0=t_0<t_1<t_2<\cdots<t_m=t$. The equation of value at time t for this cashflow is

$$V(t)=V(0)(1+i)^t+\sum_{j=1}^{n}C(t_j)(t_j-t_{j-1})(1+i)^{t-t_j},$$

which tends to

$$V(t)=V(0)(1+i)^t+\int_0^t C(s)(1+i)^{t-s}\,ds$$

as $\max_{1\le j\le n}(t_j-t_{j-1})\to 0$.
Recall that the Riemann integral of a function f is defined as

$$\int_0^t f(s)\,ds=\lim_{\max_{1\le j\le n}(t_j-t_{j-1})\to 0}\sum_{j=1}^{n}f(t_j)(t_j-t_{j-1}).$$

Example 1

A continuous–year annuity pays a constant rate 1 *at time* t *where* $0 \le t \le n$*. Interest is compounded with an annual rate of interest of* i*.*

(i) Find the present value of the annuity at time 0*.*

(ii) Find the future value of the annuity at time n*.*

Solution: (i) The present value of this continuous annuity is

$$PV = \int_0^n (1+i)^{-t}\,dt = \int_0^n e^{-t\ln(1+i)}\,dt = \frac{-e^{-t\ln(1+i)}}{\ln(1+i)}\Big|_0^n$$

$$= \frac{1}{\ln(1+i)} - \frac{e^{-n\ln(1+i)}}{\ln(1+i)} = \frac{1-(1+i)^{-n}}{\ln(1+i)}.$$

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(ii) The future value of the continuous annuity at time n is

$$FV = \int_0^n (1+i)^{n-t}\,dt = (1+i)^n PV = \frac{(1+i)^n - 1}{\ln(1+i)}.$$

If instead of compound interest, the time value of money follows the accumulation function $a(t)$, then the future value at time t of an initial outstanding balance $V(0)$ and **continuous payments** $C(s)$, in the interval $0 \le s \le t$ is

$$V(t) = V(0)a(t) + \int_0^t C(s)\frac{a(t)}{a(s)}\,ds.$$

Example 2

The force of interest at time t is $\delta_t = \frac{t^3}{10}$. Find the present value of a four–year continuous annuity which has a rate of payments at time t of $5t^3$.

Solution: The accumulation function is

$$a(t) = \exp\left(\int_0^t \delta_s \, ds\right)$$

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The present value of the four-year continuous annuity is

$$\int_0^4 \frac{C(s)}{a(s)} \, ds = \int_0^4 5s^3 e^{-\frac{s^4}{40}} \, ds = -50e^{\frac{-s^4}{40}} \Big|_0^4 = 50 - 50e^{-6.4}$$

$=49.91692.$

