Manual for SOA Exam FM/CAS Exam 2.

Chapter 3. Annuities.

Section 3.1. Geometric series.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

We use the summation notation $\sum_{i=m}^{n} x_i$ to mean

$$x_m + x_{m+1} + \cdots + x_{n-1} + x_n.$$

Usually arithmetic rules hold. In particular:

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- $\sum_{i=m}^{n} 1 = n - m + 1$.

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$$x_{n-1} + d = a + (n-1)d + d = a + nd = x_n.$$

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If a sequence $\{x_n\}_{n=0}^{\infty}$ *of real numbers satisfies* $x_n = x_{n-1} + d$, *for each* $n \geq 1$, *then* $x_n = x_0 + nd$ *for each* $n \geq 1$.

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Proof.

The proof is by induction on n. The case $n = 0$ is obvious. Assume that the case n holds. Then,

$x_{n+1} = x_n + d = x_0 + nd + d = x_0 + (n+1)d$. □

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Proof.

$$2\sum_{j=1}^{n} j = (1+2+\cdots+(n-1)+n)+(1+2+\cdots+(n-1)+n)$$
$$=(1+2+\cdots+(n-1)+n)+(n+(n-1)+\cdots+2+1)$$
$$=(1+n)+(2+n-1)+(3+n-2)\cdots+(n+1)$$
$$=(n+1)+(n+1)+(n+1)\cdots+(n+1)=n(n+1).$$

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Previous theorem can be proved by induction.
Note that in the summation $\sum_{j=1}^{n} j$, there are n numbers and the average of these numbers is $\frac{n+1}{2}$. Hence, $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.

For an arithmetic sequence,

$$\sum_{j=0}^{n}(a+jd)=\sum_{j=0}^{n}a+d\sum_{j=0}^{n}j=(n+1)a+d\frac{n(n+1)}{2}.$$

Example 1

Find $\sum_{k=10}^{100} k$.

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Solution 1:

$\sum_{k=10}^{100} k = \sum_{k=1}^{100} k - \sum_{k=1}^{9} k = \frac{(100)(101)}{2} - \frac{(9)(10)}{2} = 5005.$

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Solution 1:
$\sum_{k=10}^{100} k = \sum_{k=1}^{100} k - \sum_{k=1}^{9} k = \frac{(100)(101)}{2} - \frac{(9)(10)}{2} = 5005.$

Solution 2: By the change of variables $k = j + 9$,

$$\sum_{k=10}^{100} k = \sum_{j=1}^{91} (j + 9) = \frac{(91)(92)}{2} + (9)(91) = 5005.$$

Notice that if $k = 10$, then $j = 9$; and if $k = 100$, then $j = 91$.

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The geometric sequence $\{ar^n\}_{n=0}^{\infty}$ satisfies that for each $n \geq 1$, $rx_{n-1} = rar^{n-1} = ar^n = x_n$, where $x_n = ar^n$.

Theorem 3

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Proof.

The proof is by induction on n. The case $n = 0$ is obvious. Assume that the case n holds. Then,
$x_{n+1} = rx_n = rx_0 r^n = x_0 r^{n+1} = x_{n+1}$. □

Theorem 4

For any $r \in \mathbb{R}$,

$$r^{n+1} - 1 = (r-1)\sum_{j=0}^{n} r^j.$$

Proof.

$$\sum_{j=0}^{n} r^j(r-1) = (1 + r + r^2 + \cdots + r^n)(r-1)$$
$$= (r + r^2 + \cdots + r^{n+1}) - (1 + r + r^2 + \cdots + r^n) = r^{n+1} - 1.$$

□

In particular, we that

$$\begin{aligned}
(x^2-1) &= (x-1)(1+x), \\
(x^3-1) &= (x-1)(1+x+x^2), \\
(x^4-1) &= (x-1)(1+x+x^2+x^3), \\
(x^5-1) &= (x-1)(1+x+x^2+x^3+x^4).
\end{aligned}$$

Corollary 1

(i) If $r \neq 1$, $\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$.
(ii) If $r = 1$, $\sum_{j=0}^{n} r^j = n+1$.

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Proof.

(i) If $r \neq 1$, from $r^{n+1} - 1 = (r-1)\sum_{j=0}^{n} r^j$, we get that

$$\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}.$$

(ii) If $r = 1$,

$$\sum_{j=0}^{n} r^j = \sum_{j=0}^{n} 1 = n+1.$$

□

Example 2

Find $\sum_{k=5}^{20} 2^k$.

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Solution:

$$\sum_{k=5}^{20} 2^k = \sum_{k=0}^{20} 2^k - \sum_{k=0}^{4} 2^k = \frac{2^{21}-1}{2-1} - \frac{2^5-1}{2-1} = 2^{21} - 2^5 = 2097120.$$

We also have that for $1 \le m \le n$ and $r \neq 1$,

$$\sum_{j=m}^{n} r^j = \sum_{j=0}^{n} r^j - \sum_{j=0}^{m-1} r^j = \frac{r^{n+1}-1}{r-1} - \frac{r^m - 1}{r-1} = \frac{r^{n+1}-r^m}{r-1}.$$

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Find $\sum_{k=5}^{20} 2^k$.

Solution: $\sum_{k=5}^{20} 2^k = \frac{2^{21}-2^5}{2-1} = 2^{21} - 2^5 = 2097120$.

Theorem 5

For $|r| < 1$, $\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$.

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Proof.

If $|r| < 1$, then $\ln(|r|) < 0$ and $|r^{n+1}| \le |r|^{n+1} = e^{(n+1)\ln(|r|)} \to 0$, as $n \to \infty$. Hence,

$$\sum_{j=0}^{\infty} r^j = \lim_{n\to\infty} \sum_{j=0}^{n} r^j = \lim_{n\to\infty} \frac{r^{n+1}-1}{r-1} = \frac{1}{1-r}.$$

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Proof.

By the change of variables $j = k + n$,

$$\sum_{j=n}^{\infty} r^j = \sum_{k=0}^{\infty} r^{k+n} = r^n \sum_{k=0}^{\infty} r^k = \frac{r^n}{1-r}.$$

□

Example 4

Find $\sum_{k=9}^{\infty} 3^{-k}$.

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Solution:

$$\sum_{k=9}^{\infty} 3^{-k} = \frac{3^{-9}}{1-(1/3)} = 0.0000762079.$$

Theorem 6

For $r \neq 1$,

$$\sum_{j=1}^{n} jr^j = \frac{nr^{n+2} - (n+1)r^{n+1} + r}{(r-1)^2}.$$

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Proof.

Taking derivatives with respect to r in the inequality $\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$, we get that

$$\sum_{j=1}^{n} jr^{j-1} = \frac{(n+1)r^n(r-1)-(r^{n+1}-1)}{(r-1)^2} = \frac{nr^{n+1}-(n+1)r^n+1}{(r-1)^2}$$

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Proof.

If $|r| < 1$, then

$$\sum_{j=1}^{\infty} jr^j = \lim_{n\to\infty} \sum_{j=1}^{n} jr^j = \lim_{n\to\infty} \frac{nr^{n+2} - (n+1)r^{n+1} + r}{(r-1)^2} = \frac{r}{(r-1)^2}.$$

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Solution:

$$\sum_{k=1}^{\infty} k4^{-k} = \frac{4^{-1}}{(1-4^{-1})^2} = \frac{4}{9} = 0.4444444444.$$

