Manual for SOA Exam FM/CAS Exam 2.

Chapter 3. Annuities.

Section 3.2. Level payment annuities.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2,
Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

An **annuity** is a sequence of payments made at equal intervals of time. We have n periods of times $[0, t], [t, 2t], [2t, 3t], \dots [(n-1)t, nt]$ with the same length. By a change of units, we will assume that intervals have unit length. So, the intervals are $[0, 1], [1, 2], [2, 3], \dots [n-1, n]$. We order the periods as follows:

Time interval	Name	Beginning of the period	End of the period
$[0, 1]$	1st period	time 0	time 1
$[1, 2]$	2nd period	time 1	time 2
$[2, 3]$	3rd period	time 2	time 3
$\dots$	$\dots$	$\dots$	$\dots$
$[n-2, n-1]$	$(n-1)$–th period	time $n-2$	time $n-1$
$[n-1, n]$	n–th period	time $n-1$	time n

An annuity is said to have **level payments** if all payments C_j are equal.

An annuity is said to have **level payments** if all payments C_j are equal.
An annuity has **non–level payments** if some payments C_j are different from other ones.

An annuity is said to have **level payments** if all payments C_j are equal.
An annuity has **non–level payments** if some payments C_j are different from other ones.
The payments can be made either at the beginning or at the end of intervals of time.

An annuity is said to have **level payments** if all payments C_j are equal.

An annuity has **non–level payments** if some payments C_j are different from other ones.

The payments can be made either at the beginning or at the end of intervals of time.

For an **annuity–immediate** payments are made at the end of the intervals of time.

An annuity–immediate is a cashflow of the type:

Contributions	0	C_1	C_2	$\cdots$	C_n
Time	0	1	2	$\cdots$	n

An annuity is said to have **level payments** if all payments C_j are equal.

An annuity has **non–level payments** if some payments C_j are different from other ones.

The payments can be made either at the beginning or at the end of intervals of time.

For an **annuity–immediate** payments are made at the end of the intervals of time.

An annuity–immediate is a cashflow of the type:

Contributions	0	C_1	C_2	$\cdots$	C_n
Time	0	1	2	$\cdots$	n

For an **annuity–due** the payments are made at the beginning of the intervals of time.

An annuity–due is a cashflow of the type:

Contributions	C_0	C_1	$\cdots$	C_{n-1}	0
Time	0	1	$\cdots$	$n-1$	n

The cashflow of an annuity–immediate with level payments of one is

Contributions	0	1	1	$\cdots$	1
Time	0	1	2	$\cdots$	n

If the time value of the money follows an accumulation function $a(t)$, then the **present value of an annuity–immediate with level annual payments of one** is

$$a_{\overline{n}|} = \frac{1}{a(1)} + \frac{1}{a(2)} + \cdots + \frac{1}{a(n)} = \sum_{j=1}^{n} \frac{1}{a(j)}.$$

The **accumulated value of an annuity–immediate with level annual payments of one** is

$$s_{\overline{n}|} = \frac{a(n)}{a(1)} + \frac{a(n)}{a(2)} + \cdots + \frac{a(n)}{a(n)} = \sum_{j=1}^{n} \frac{a(n)}{a(j)}.$$

Example 1

You are given that $\delta_t = \frac{1}{8+t}$, $t \geq 0$. *Find* $a_{\overline{n|}}$ *and* $s_{\overline{n|}}$.

Example 1

You are given that $\delta_t = \frac{1}{8+t}$, $t \geq 0$. *Find* $a_{\overline{n}|}$ *and* $s_{\overline{n}|}$.

Solution: We have that

$$\int_0^t \delta_s \, ds = \int_0^t \frac{1}{8+s} \, ds = \ln(8+s) \,\Big|_0^t = \ln\left(\frac{8+t}{8}\right).$$

So, $a(t) = e^{\int_0^t \delta_s} = \frac{8+t}{8}$,

$$a_{\overline{n}|} = \sum_{j=1}^{n} \frac{1}{a(j)} = \sum_{j=1}^{n} \frac{8}{8+j}$$

and

$$s_{\overline{n}|} = \sum_{j=1}^{n} \frac{a(n)}{a(j)} = \sum_{j=1}^{n} \frac{8+n}{8+j}.$$

The cashflow of an annuity–due with n level payments of one is

Contributions	1	1	1	$\cdots$	1	0
Time	0	1	2	$\cdots$	$n-1$	n

The **present value of an annuity–due with n level annual payments of one** is

$$\ddot{a}_{\overline{n|}} = 1 + \frac{1}{a(1)} + \frac{1}{a(2)} + \cdots + \frac{1}{a(n-1)} = \sum_{j=0}^{n-1} \frac{1}{a(j)}.$$

The **future value at time n of an annuity–due with level annual payments of one** is

$$\ddot{s}_{\overline{n|}} = a(n) + \frac{a(n)}{a(1)} + \frac{a(n)}{a(2)} + \cdots + \frac{a(n)}{a(n-1)} = \sum_{j=0}^{n-1} \frac{a(n)}{a(j)}.$$

Example 2

Suppose that the annual effective interest rate for year n is $i_n = \frac{2}{n+4}$. *Find* $\ddot{a}_{\overline{n|}}$ *and* $\ddot{s}_{\overline{n|}}$.

Example 2

Suppose that the annual effective interest rate for year n is $i_n = \frac{2}{n+4}$*. Find* $\ddot{a}_{\overline{n}|}$ *and* $\ddot{s}_{\overline{n}|}$*.*

Solution: Since $1 + i_n = 1 + \frac{2}{n+4} = \frac{n+6}{n+4}$,

$$a(n) = (1+i_1)(1+i_1)\cdots(1+i_n) = \frac{1+6}{1+4}\cdot\frac{2+6}{2+4}\cdot\frac{3+6}{3+4}\cdots\frac{n+6}{n+4}$$

$$=\frac{(n+6)(n+5)}{30},$$

$$\ddot{a}_{\overline{n}|} = \sum_{j=1}^{n}\frac{1}{a(j)} = \sum_{j=0}^{n-1}\frac{30}{(j+6)(j+5)} = \sum_{j=0}^{n-1}(30)\left(\frac{1}{j+5}-\frac{1}{j+6}\right)$$

$$=30\left(\frac{1}{0+5}+\cdots+\frac{1}{n-1+5}\right) - 30\left(\frac{1}{1+5}+\cdots+\frac{1}{n+5}\right)$$

$$=(30)\left(\frac{1}{5}-\frac{1}{n+5}\right) = \frac{6n}{n+5}, \quad \ddot{s}_{\overline{n}|} = a(n)\ddot{a}_{\overline{n}|} = \frac{n(n+6)}{5}.$$

Remember:
The cashflow of an annuity–immediate with n level payments of one is

Contributions	0	1	1	$\cdots$	1
Time	0	1	2	$\cdots$	n

Remember:
The cashflow of an annuity–immediate with n level payments of one is

Contributions	0	1	1	$\cdots$	1
Time	0	1	2	$\cdots$	n

The cashflow of an annuity–due with n level payments of one is

Contributions	1	1	1	$\cdots$	1	0
Time	0	1	2	$\cdots$	$n-1$	n

Theorem 1

If $i \neq 0$, the **present value of an annuity–immediate with level payments of one** *is*

$$a_{\overline{n|}i} = a_{\overline{n|}} = \nu + \nu^2 + \cdots + \nu^n = \frac{\nu(1-\nu^n)}{1-\nu} = \frac{1-\nu^n}{\frac{1}{\nu}-1} = \frac{1-\nu^n}{i}$$

$$= \frac{1-(1+i)^{-n}}{i},$$

where we have used that $\nu(1+i) = 1$.

Theorem 1

If $i \neq 0$*, the* **present value of an annuity–immediate with level payments of one** *is*

$$a_{\overline{n}|i} = a_{\overline{n}|} = \nu + \nu^2 + \cdots + \nu^n = \frac{\nu(1-\nu^n)}{1-\nu} = \frac{1-\nu^n}{\frac{1}{\nu}-1} = \frac{1-\nu^n}{i}$$
$$= \frac{1-(1+i)^{-n}}{i},$$

where we have used that $\nu(1+i) = 1$*.*

If $i \neq 0$*, the* **future value of an annuity–immediate with level payments of one** *at time* n *is*

$$s_{\overline{n}|i} = s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1 = \frac{(1+i)^n - 1}{i}.$$

Theorem 1

If $i \neq 0$*, the* **present value of an annuity–immediate with level payments of one** *is*

$$a_{\overline{n|}i} = a_{\overline{n|}} = \nu + \nu^2 + \cdots + \nu^n = \frac{\nu(1-\nu^n)}{1-\nu} = \frac{1-\nu^n}{\frac{1}{\nu}-1} = \frac{1-\nu^n}{i}$$

$$= \frac{1-(1+i)^{-n}}{i},$$

where we have used that $\nu(1+i) = 1$.

If $i \neq 0$*, the* **future value of an annuity–immediate with level payments of one** *at time* n *is*

$$s_{\overline{n|}i} = s_{\overline{n|}} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1 = \frac{(1+i)^n - 1}{i}.$$

If $i = 0$, $a_{\overline{n|}i} = s_{\overline{n|}i} = n$.

For example,

$$a_{\overline{1|}i} = \nu,$$
$$a_{\overline{2|}i} = \nu + \nu^2,$$
$$a_{\overline{3|}i} = \nu + \nu^2 + \nu^3,$$
$$a_{\overline{4|}i} = \nu + \nu^2 + \nu^3 + \nu^4.$$

$$s_{\overline{1|}i} = 1,$$
$$s_{\overline{2|}i} = 1 + 1 + i,$$
$$s_{\overline{3|}i} = 1 + 1 + i + (1+i)^2,$$
$$s_{\overline{4|}i} = 1 + 1 + i + (1+i)^2 + (1+i)^3.$$

Example 3

Calculate the present value of \$5000 paid at the end of each year for 15 years using an annual effective interest rate of 7.5%.

Example 3

Calculate the present value of $5000 paid at the end of each year for 15 years using an annual effective interest rate of 7.5%.

Solution: The present value is

$$(5000)a_{\overline{15|}0.075} = (5000)\frac{1-(1.075)^{-15}}{0.075} = 44135.59873.$$

If every period is exactly one year, then i in the formulas above is the annual effective rate of interest. If the length of a period is not a year, i in the formulas above is the effective rate of interest per period. If each period lasts t years, then the t–year interest factor is $(1+i)^t$ and the t–year effective rate of interest is $(1+i)^t - 1$. So, if each period lasts t years, we use the previous formulas with i replaced by $(1+i)^t - 1$, where the last i is the annual effective rate of interest.

For example, suppose that the payments are made each $1/m$ years. Then, the interest factor for $1/m$ years is $(1+i)^{1/m}$ and the $1/m$ year interest rate is $(1+i)^{1/m} - 1 = \frac{i^{(m)}}{m}$, where $i^{(m)}$ is the nominal annual rate of interest convertible m times a year. For example, the present value at time 0 of the annuity

Contributions	0	1	1	$\cdots$	1
Time (in years)	0	$1/m$	$2/m$	$\cdots$	n/m

is $a_{\overline{n}|(1+i)^{1/m}-1} = a_{\overline{n}|i^{(m)}/m}$. The future value at time n/m years of the previous cashflow is $s_{\overline{n}|(1+i)^{1/m}-1} = s_{\overline{n}|i^{(m)}/m}$.

Example 4

John invest \$500 into an account at the end of each month for 5 years. The annual effective interest rate is 4.5%. Calculate the balance of this account at the end of 5 years.

Example 4

John invest \$500 into an account at the end of each month for 5 years. The annual effective interest rate is 4.5%. Calculate the balance of this account at the end of 5 years.

Solution: The number of payments made is $(5)(12) = 60$. The cashflow is

Contributions	500	500	500	$\cdots$	500
Time (in months)	1	2	2	$\cdots$	60

The one–month effective interest rate is $(1.045)^{1/12} - 1$. Hence, the balance of this account at the end of 5 years is

$$(500)s_{\overline{60}|(1.045)^{1/12}-1} = (500)\frac{((1.045)^{1/12} - 1 + 1)^{60} - 1}{(1.045)^{1/12} - 1}$$

$$=(500)\frac{(1.045)^{5} - 1}{(1.045)^{1/12} - 1} = 33495.8784.$$

If payments are made every m years, the interest factor per period is $(1+i)^m - 1$. For example, the future value at time nm years of the annuity

Contributions	0	1	1	$\cdots$	1
Time (in years)	0	m	$2m$	$\cdots$	nm

is $a_{\overline{n}|(1+i)^m-1}$, where i is the annual effective rate of interest. The future value at time nm years of the previous annuity is $s_{\overline{n}|(1+i)^m-1}$.

Example 5

A cashflow pays $8000 at the end of every other year for 16 years. The first payment is made in two years. The annual effective interest rate is 6.5%. Calculate the present value of this cashflow.

Example 5

A cashflow pays \$8000 at the end of every other year for 16 years. The first payment is made in two years. The annual effective interest rate is 6.5%. Calculate the present value of this cashflow.

Solution: The cashflow is

Contributions	8000	8000	8000	$\cdots$	8000
Time (in years)	2	4	6	$\cdots$	16

Notice that eight payments are made. The two–year effective interest rate is $(1.065)^2 - 1$. Hence, the present value of the cashflow is

$$(8000)a_{\overline{8}|(1.065)^2-1} = (8000)\frac{1-((1.065)^2-1+1)^{-8}}{(1.065)^2-1}$$

$$=(8000)\frac{1-(1.065)^{-16}}{(1.065)^2-1} = 37841.21717.$$

Theorem 2

If $i \neq 0$, *the* **present value of an annuity–due with level payments of one** *is*

$$\ddot{a}_{\overline{n}|i} = \ddot{a}_{\overline{n}|} = 1 + \nu + \nu^2 + \cdots + \nu^{n-1} = \frac{1-\nu^n}{1-\nu} = \frac{1-\nu^n}{d},$$

where we have used that $\nu = 1 - d$.
If $i \neq 0$, *the* **future value at time** n **of an annuity–due with level payments of one** *is*

$$\ddot{s}_{\overline{n}|i} = \ddot{s}_{\overline{n}|i} = (1+i)^n + (1+i)^{n-1} + \cdots + (1+i)$$
$$= \frac{(1+i)^{n+1} - (1+i)}{i} = \frac{(1+i)^n - 1}{d},$$

where we have used that $1 - d = \frac{1}{1+i}$ *and* $d = \frac{i}{1+i}$.
If $i = 0$, $\ddot{a}_{\overline{n}|i} = \ddot{s}_{\overline{n}|i} = n$.

The annuities factors which have introduced give the present value of the cashflow

Contributions	1	1	$\cdots$	1
Time	1	2	$\cdots$	n

at different times.
The present value of the cashflow at time 0 is $a_{\overline{n}|i}$.
The present value of the cashflow at time 1 is $\ddot{a}_{\overline{n}|i}$.
The present value of the cashflow at time n is $s_{\overline{n}|i}$.
The present value of the cashflow at time $n+1$ is $\ddot{s}_{\overline{n}|i}$.

Figure 1: Present value of an annuity at different times

Often, the contributions do not start at time 0. But, $a_{\overline{n}|i}$ is always the present value of a level unity annuity one period before the first payment. $\ddot{a}_{\overline{n}|i}$ is the present value of a level unity annuity at the time of the first payment. $s_{\overline{n}|i}$ is the future value of a level unity annuity at the time of the last payment. $\ddot{s}_{\overline{n}|i}$ is the future value of a level unity annuity one period after the last payment. For example, for the cashflow

Contributions	1	1	$\cdots$	1
Time	$t+1$	$t+2$	$\cdots$	$t+n$

The present value of the cashflow at time t is $a_{\overline{n}|i}$.
The present value of the cashflow at time $t+1$ is $\ddot{a}_{\overline{n}|i}$.
The present value of the cashflow at time $t+n$ is $s_{\overline{n}|i}$.
The present value of the cashflow at time $t+n+1$ is $\ddot{s}_{\overline{n}|i}$.

Figure 2: Present value of an annuity at different times

Example 6

Investment contributions of $4500 are made at the beginning of the year for 20 years into an account. This account pays an annual effective interest rate is 7%. Calculate the accumulated value at the end of 25 years.

Solution: The cashflow of payments is

Contributions	4500	4500	$\cdots$	4500
Time	0	1	$\cdots$	19

We can find the accumulated value at the end of 25 years doing

Example 6

Investment contributions of \$4500 are made at the beginning of the year for 20 years into an account. This account pays an annual effective interest rate is 7%. Calculate the accumulated value at the end of 25 years.

Solution: The cashflow of payments is

Contributions	4500	4500	$\cdots$	4500
Time	0	1	$\cdots$	19

We can find the accumulated value at the end of 25 years doing either

$$a_{\overline{20}|0.07}(1+0.07)^{26} = \frac{1-(1+0.07)^{-20}}{0.07}(1+0.07)^{26}$$
$$=\frac{(1+0.07)^{26}-(1+0.07)^{6}}{0.07} = 276854.31,$$

Example 6

Investment contributions of \$4500 are made at the beginning of the year for 20 years into an account. This account pays an annual effective interest rate is 7%. Calculate the accumulated value at the end of 25 years.

Solution: The cashflow of payments is

Contributions	4500	4500	$\cdots$	4500
Time	0	1	$\cdots$	19

We can find the accumulated value at the end of 25 years doing or

$$\ddot{a}_{\overline{20}|0.07}(1+0.07)^{25} = \frac{1-(1+0.07)^{-20}}{1-(1+0.07)^{-1}}(1+0.07)^{25}$$

$$= \frac{(1+0.07)^{26}-(1+0.07)^{6}}{0.07} = 276854.31,$$

Example 6

Investment contributions of $4500 are made at the beginning of the year for 20 years into an account. This account pays an annual effective interest rate is 7%. Calculate the accumulated value at the end of 25 years.

Solution: The cashflow of payments is

Contributions	4500	4500	$\cdots$	4500
Time	0	1	$\cdots$	19

We can find the accumulated value at the end of 25 years doing or

$$s_{\overline{20}|0.07}(1+0.07)^6 = \frac{(1+0.07)^{20}-1}{0.07}(1+0.07)^6$$
$$= \frac{(1+0.07)^{26}-(1+0.07)^6}{0.07} = 276854.31,$$

Example 6

Investment contributions of $4500 are made at the beginning of the year for 20 years into an account. This account pays an annual effective interest rate is 7%. Calculate the accumulated value at the end of 25 years.

Solution: The cashflow of payments is

Contributions	4500	4500	$\cdots$	4500
Time	0	1	$\cdots$	19

We can find the accumulated value at the end of 25 years doing or

$$\ddot{s}_{\overline{20}|0.07}(1+0.07)^5 = \frac{(1+0.07)^{20}-1}{1-(1+0.07)^{-1}}(1+0.07)^5$$

$$=\frac{(1+0.07)^{26}-(1+0.07)^6}{0.07} = 276854.31.$$

Theorem 3

(Interest factor relations for annuities)

$$\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}, \quad s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i}, \quad \ddot{s}_{\overline{n}|i} = (1+i)^{n+1} a_{\overline{n}|i}$$

Theorem 3

(Interest factor relations for annuities)

$$\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}, \quad s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i}, \quad \ddot{s}_{\overline{n}|i} = (1+i)^{n+1} a_{\overline{n}|i}$$

Proof.

Consider the cashflow:

Contributions	0	1	1	$\cdots$	1
Time	0	1	2	$\cdots$	n

The present value at time 0 of the cashflow is $a_{\overline{n}|i}$.
The present value at time 1 of the cashflow is $\ddot{a}_{\overline{n}|i}$.
The present value at time n of the cashflow is $s_{\overline{n}|i}$.
The present value at time $n+1$ of the cashflow is $\ddot{s}_{\overline{n}|i}$.

□

An application of

$$\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}.$$

Example 7

If $a_{\overline{n}|i} = 11.5174109$ *and* $\ddot{a}_{\overline{n}|i} = 11.9205203$, *find* n.

An application of

$$\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}.$$

Example 7

If $a_{\overline{n}|i} = 11.5174109$ *and* $\ddot{a}_{\overline{n}|i} = 11.9205203$*, find* n*.*

Solution: We have that $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$. So, $1+i = \frac{11.9205203}{11.5174109} = 1.035$ and $i = 3.5\%$. Solving $11.5174109 = a_{\overline{n}|3.5\%} = \frac{1-(1.035)^{-n}}{0.035}$, we get $(1.035)^{-n} = 1-(11.5174109)(0.035) = 0.5968906185$ and $n = \frac{-\log 0.5968906185}{\log 1.035} = 15$.

An application of

$$\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}.$$

Example 8

If $\ddot{s}_{\overline{n}|i} = 21$ *and* $s_{\overline{n}|i} = 20$*, find* i*.*

An application of

$$\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}.$$

Example 8

If $\ddot{s}_{\overline{n}|i} = 21$ *and* $s_{\overline{n}|i} = 20$, *find* i.

Solution: We have that

$$21 = \ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i} = (1+i)(20).$$

So, $i = \frac{21}{20} - 1 = 5\%$.

Theorem 4
(Induction relations)

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i} \quad \text{and} \quad s_{\overline{n}|i} = 1 + \ddot{s}_{\overline{n-1}|i}.$$

Theorem 4

*(***Induction relations***)*

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i} \quad \text{and} \quad s_{\overline{n}|i} = 1 + \ddot{s}_{\overline{n-1}|i}.$$

Proof.

Consider the cashflows

Cashflow 1	Contributions	1	0	0	$\cdots$	$\cdots$	0	0
Cashflow 2	Contributions	0	1	1	$\cdots$	$\cdots$	1	0
Cashflow 3	Contributions	1	1	1	$\cdots$	$\cdots$	1	0
	Time	0	1	2	$\cdots$	$\cdots$	$n-1$	n

We have that the third cashflow is the sum of the first two. The present value of the previous cashflows are 1 and $a_{\overline{n-1}|i}$ and $\ddot{a}_{\overline{n}|i}$, respectively. This implies the first relation. The second relation follows similarly. $\square$

An application of

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}.$$

Example 9

If $\ddot{a}_{\overline{10}|i} = 8$, *find* $a_{\overline{9}|i}$.

An application of

$$\ddot{a}_{\overline{n}|i} = 1 + a_{\overline{n-1}|i}.$$

Example 9

If $\ddot{a}_{\overline{10}|i} = 8$, *find* $a_{\overline{9}|i}$.

Solution: We have that $a_{\overline{9}|i} = \ddot{a}_{\overline{10}|i} - 1 = 7$.

An application of

$$s_{\overline{n}|i} = 1 + \ddot{s}_{\overline{n-1}|i}$$

Example 10

If $\ddot{s}_{\overline{10}|i} = 15$, *find* $s_{\overline{11}|i}$.

An application of

$$s_{\overline{n}|i} = 1 + \ddot{s}_{\overline{n-1}|i}$$

Example 10

If $\ddot{s}_{\overline{10}|i} = 15$, *find* $s_{\overline{11}|i}$.

Solution: We have that $s_{\overline{11}|i} = 1 + \ddot{s}_{\overline{10}|i} = 16$.

Theorem 5
(Amortization relations)

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i \quad \text{and} \quad \frac{1}{\ddot{a}_{\overline{n}|i}} = \frac{1}{\ddot{s}_{\overline{n}|i}} + d.$$

Proof of $\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i$: Suppose that a loan of \$1 is paid in n payments made at the end of the each period. Then, each payment should be $\frac{1}{a_{\overline{n}|i}}$. Suppose that we pay the loan as follows, we pay i at the end of each period and put x in an extra account paying an effective rate of interest i. Since the initial loan is \$1 the interest accrued at the end of the first period is i. But, we pay i at the end of the first period. Hence, immediately after this payment we owe \$1. Proceeding in this way, we deduce that we owe \$1 immediately after each payment. The money in the extra account accumulates to $xs_{\overline{n}|i}$ at time n. In order to pay the loan, we need $x = \frac{1}{s_{\overline{n}|i}}$. Our total payments at the end of the each period are $\frac{1}{s_{\overline{n}|i}} + i$. Since this series of payments repays the loan of \$1, we must have that $\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i$. The proof of the second formula is similar and it is omitted.

An application of

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i.$$

Example 11

If $s_{\overline{n}|i} = 15.9171265$ *and* $a_{\overline{n}|i} = 8.86325164$, *calculate* n.

An application of

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i.$$

Example 11

If $s_{\overline{n}|i} = 15.9171265$ *and* $a_{\overline{n}|i} = 8.86325164$, *calculate* n.

Solution: Using that $\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i$,we get that $i = \frac{1}{8.86325164} - \frac{1}{15.9171265} = 5\%$. From $a_{\overline{n}|5\%} = 8.86325164$, we get $n = 12$.

In the calculator TI–BA–II–Plus we can use the time value of money worksheet to solve problems with annuities. There are 5 main financial variables in this worksheet:

- The number of periods $\boxed{\text{N}}$.
- The nominal interest for year $\boxed{\text{I/Y}}$.
- The present value $\boxed{\text{PV}}$.
- The payment per period $\boxed{\text{PMT}}$.
- The future value $\boxed{\text{FV}}$.

Recall that how to use the money worksheet was explained in Section 1.3. It is **recommended** that you set–up $\boxed{\text{P/Y}}$=1 and $\boxed{\text{C/Y}}$=1, by pressing:
$\boxed{\text{2nd}}$, $\boxed{\text{P/Y}}$, $\boxed{1}$, $\boxed{\text{ENTER}}$, $\boxed{\downarrow}$, $\boxed{1}$, $\boxed{\text{ENTER}}$, $\boxed{\text{2nd}}$, $\boxed{\text{QUIT}}$.
Unless it is said otherwise, we will **assume that the entries for $\boxed{\text{C/Y}}$ and $\boxed{\text{P/Y}}$ are both 1**. To check that this is so, do

$$\boxed{\text{2nd}}\ \boxed{\text{P/Y}}\ \boxed{\downarrow}\ \boxed{\text{2nd}}\ \boxed{\text{QUIT}}.$$

When BGN is set–up at END (and $\boxed{\text{C/Y}}$ and $\boxed{\text{P/Y}}$ have value 1), the value time of money formula in the calculator is

$$PV + PMT \cdot \frac{1-(1+i)^{-N}}{i} + FV(1+i)^{-N} = 0.$$

Using the calculator, we can solve for any variable in the equation:

$$L + Pa_{\overline{n}|i} + F(1+i)^{-n} = 0.$$

This equation is equivalent to

$$L(1+i)^{n} + Ps_{\overline{n}|i} + F = 0.$$

In the calculator, we input the variables we know using: L as the $\boxed{\text{PV}}$, P as the $\boxed{\text{PMT}}$, F as the $\boxed{\text{FV}}$, i as the $\boxed{\text{I/Y}}$, n as the $\boxed{\text{N}}$. L, P and F can take negative values.

To solve for any variable from the equation

$$L + Pa_{\overline{n}|i} = 0$$

we input in the calculator: L as $\boxed{\text{PV}}$, P as $\boxed{\text{PMT}}$, 0 as $\boxed{\text{FV}}$, i as the $\boxed{\text{I/Y}}$, n as $\boxed{\text{N}}$. If we are solving for either $\boxed{\text{I/Y}}$ or $\boxed{\text{N}}$, $\boxed{\text{PV}}$ and $\boxed{\text{PMT}}$ must have different signs.

To solve for any variable from the equation

$$Ps_{\overline{n}|i} + F = 0.$$

we input: F as the $\boxed{\text{FV}}$, P as the $\boxed{\text{PMT}}$, i as the $\boxed{\text{I/Y}}$, n as the $\boxed{\text{N}}$ and 0 as the $\boxed{\text{PV}}$. If we are solving for either $\boxed{\text{I/Y}}$ or $\boxed{\text{N}}$, $\boxed{\text{FV}}$ and $\boxed{\text{PMT}}$ must have different signs.

Example 12

What must you deposit at the end of each of the next 10 years in order to accumulate 20,000 at the end of the 10 years assuming $i = 5\%$.

Example 12

What must you deposit at the end of each of the next 10 years in order to accumulate 20,000 at the end of the 10 years assuming $i = 5\%$.

Solution: We solve for P in $20000 = Ps_{\overline{10}|0.05}$ and get $P = 1590.091499$. In the calculator TI–BA–II–Plus, press:
$\boxed{10}$ $\boxed{\text{N}}$ $\boxed{20000}$ $\boxed{\text{FV}}$ $\boxed{5}$ $\boxed{\text{I/Y}}$ $\boxed{0}$ $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PMT}}$
Note the display in the calculator is negative.

Example 13

If $i^{(12)} = 9\%$ and \$300 is deposited at the end of the each month for 1 year, what will the accumulated value be in 1 year?

Example 13

If $i^{(12)} = 9\%$ and \$300 is deposited at the end of the each month for 1 year, what will the accumulated value be in 1 year?

Solution: We solve for FV in $FV = 300 s_{\overline{12}|0.09/12}$ and get $FV = 3752.275907$. In the calculator, press:

$\boxed{12}$ $\boxed{\text{N}}$ $\boxed{300}$ $\boxed{\text{PMT}}$ $\boxed{0.75}$ $\boxed{\text{I/Y}}$ $\boxed{0}$ $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$

where we used that $9/12 = 0.75$.

Since the length of a period is a month:

- The monthly effective interest rate is $\frac{i^{(12)}}{12} = \frac{0.09}{12} = 0.75\%$.
- The number of periods is 12 months.

When BGN is set–up at BGN, the value time of money formula in this calculator is

$$PV + PMT \cdot (1+i) \cdot \frac{1-(1+i)^{-N}}{i} + FV(1+i)^{-N} = 0.$$

To solve for a variable from the equation:

$$L + P\ddot{a}_{\overline{n}|i} + F(1+i)^{-n} = 0,$$

we proceed as before, with payments set–up at beginning.
Previous equation is equivalent to

$$L(1+i)^n + P\ddot{s}_{\overline{n}|i} + F = 0,$$

To change the setting of the payments (either at the beginning of the period, or at the end of the period), press:

$\boxed{\text{2nd}}$, $\boxed{\text{BGN}}$, $\boxed{\text{2nd}}$, $\boxed{\text{SET}}$, $\boxed{\text{2nd}}$, $\boxed{\text{Quit}}$.

If the calculator is set–up with payments at the end of the periods, there is no indicator in the screen. If the calculator is set–up with payments at the beginning of the periods, the indicator "BGN" appears in the screen.

Example 14

A company purchases 100 acres of land for \$200,000 and agrees to remit 10 equal annual installments of \$27,598 each at the beginning of the year. What is the annual interest rate on this loan?

Example 14

A company purchases 100 acres of land for \$200,000 and agrees to remit 10 equal annual installments of \$27,598 each at the beginning of the year. What is the annual interest rate on this loan?

Solution: We solve for i in the equation $200000 = 27598\ddot{a}_{\overline{10}|i}$ to get $i = 8\%$. In the calculator, set payments at the beginning of the period and press:

$\boxed{10}$ $\boxed{\text{N}}$ $\boxed{-27598}$ $\boxed{\text{PMT}}$ $\boxed{200000}$ $\boxed{\text{PV}}$ $\boxed{0}$ $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$

Example 15

An annuity pays \$7000 at the end of the year for 7 years with the first payment made 5 years from now. The effective annual rate of interest is 6.5%. Find the present value of the this annuity.

Example 15

An annuity pays \$7000 at the end of the year for 7 years with the first payment made 5 years from now. The effective annual rate of interest is 6.5%. Find the present value of the this annuity.

Solution 1: The cashflow of payments is

Payments	7000	7000	7000	7000	7000	7000	7000
Time	5	6	7	8	9	10	11

Using an immediate annuity, $(7000)a_{\overline{7}|0.06}$ is the present value of the annuity, one period before the first payment, i.e. $(7000)a_{\overline{7}|0.06}$ is the present value of the annuity at time 4. So, the present value of the annuity is

$$(1.06)^{-4}(7000)a_{\overline{7}|0.06} = (0.7920937)(7000)(5.582381)$$
$$=30952.38.$$

Example 15

An annuity pays \$7000 at the end of the year for 7 years with the first payment made 5 years from now. The effective annual rate of interest is 6.5%. Find the present value of the this annuity.

Solution 2: The cashflow of payments is

Payments	7000	7000	7000	7000	7000	7000	7000
Time	5	6	7	8	9	10	11

Using a due annuity, $(7000)\ddot{a}_{\overline{7}|0.06}$ is the present value of the annuity, at the time of the first payment, i.e. $(7000)\ddot{a}_{\overline{7}|0.06}$ is the present value of the annuity at time 5. So, the present value of the annuity is

$$(1.06)^{-5}(7000)\ddot{a}_{\overline{7}|0.06} = (0.7472582)(7000)(5.917324326)$$
$$=30952.38.$$

Example 16

An annuity–immediate pays \$7000 at the end of the year for 7 years. The current annual effective rate of interest is 4.5% for the first three years and 5.5% thereafter. Find the present value of this annuity.

Example 16

An annuity–immediate pays \$7000 at the end of the year for 7 years. The current annual effective rate of interest is 4.5% for the first three years and 5.5% thereafter. Find the present value of this annuity.

Solution: The cashflow is

Payments	7000	7000	7000	7000	7000	7000	7000
Time	1	2	3	4	5	6	7

Consider two cashflows: one with the first three payments and another one with the last four payments. The present value at time 0 of the first three payments is $(7000)a_{\overline{3}|0.045}$. The present value at time 3 of the last four payments is $(7000)a_{\overline{4}|0.055}$. The present value at time 0 of the last four payments is $(7000)(1.045)^{-3}a_{\overline{4}|0.055}$. The present value of the whole annuity is

$$(7000)a_{\overline{3}|0.045} + (7000)(1.045)^{-3}a_{\overline{4}|0.055}$$
$$=19242.75048 + 21500.85804 = 40743.60852.$$

Theorem 6

Consider the cashflow

Contributions	0	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\cdots$	$\frac{1}{m}$
Time (in years)	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{m}{m}$	$\frac{m+1}{m}$	$\cdots$	$\cdots$	$\frac{nm}{m}$

The present value of this cashflow is

$$a_{\overline{n}|i}^{(m)} = \frac{1-\nu^n}{i^{(m)}}.$$

The future value at time n of this cashflow is

$$s_{\overline{n}|i}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}}.$$

Proof: The present value of the cashflow is

$$\frac{1}{m} a_{\overline{nm}|i^{(m)}/m} = \frac{1}{m} \frac{1-\left(1+\frac{i^{(m)}}{m}\right)^{-mn}}{\frac{i^{(m)}}{m}} = \frac{1-\nu^n}{i^{(m)}}.$$

Example 17

Suppose that Arthur takes a mortgage for L at an annual nominal rate of interest of 7.5% compounded monthly. The loan is paid at the end of the month with level payments of \$1200 for n years. Suppose at the last minute, Arthur changes the conditions of his loan so that the payments will biweekly. The duration of the loan and the effective annual rate remain unchanged. Calculate the amount of the biweekly payment. Assume that there are 365/7 *weeks in a year.*

Solution: Let P be the biweekly payment. We have that

$$L = (12)(1200)a_{\overline{n}|i}^{(12)} = (12)(1200)\frac{1-\nu^n}{i^{(12)}}$$

and

$$L = (365/14)(P)a_{\overline{n}|i}^{(365/14)} = (365/14)(P)\frac{1-\nu^n}{i^{(365/14)}}$$

Hence,

$$\frac{(12)(1200)}{i^{(12)}} = \frac{(365/14)(P)}{i^{(365/14)}}.$$

and

$$P = \frac{(12)(1200)i^{(365/14)}}{i^{(12)}(365/14)} = \frac{(12)(1200)(0.07485856348)}{(0.075)(365/14)} = 551.2871743,$$

using that

$$i^{(365/14)} = (365/14)\left(1+\frac{0.075}{12}\right)^{12(14/365)} = 7.485856348\%.$$

Theorem 7

Consider the cashflow

Contributions	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	0
Time (in years)	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{nm-1}{m}$	$\frac{nm}{m}$

The present value of this cashflow is

$$\ddot{a}^{(m)}_{\overline{n}|i} = \frac{1-\nu^n}{d^{(m)}}.$$

The future value at time n of this cashflow is

$$\ddot{s}^{(m)}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d^{(m)}}.$$

Proof: The present value of the cashflow is

$$\frac{1}{m}\ddot{a}_{\overline{nm}|i^{(m)}/m} = \frac{1}{m}\frac{1-\left(1+\frac{i^{(m)}}{m}\right)^{-mn}}{\frac{d^{(m)}}{m}} = \frac{1-\nu^n}{d^{(m)}}.$$

Theorem 8

For integers $k, n \geq 1$, $a_{\overline{nk}|i} = a_{\overline{n}|i} \ddot{a}_{\overline{k}|(1+i)^n-1}$.

Theorem 8

For integers $k, n \geq 1$, $a_{\overline{nk}|i} = a_{\overline{n}|i}\ddot{a}_{\overline{k}|(1+i)^n-1}$.

Proof.

We have that

$$\frac{a_{\overline{nk}|i}}{a_{\overline{n}|i}} = \frac{\frac{1-\nu^{nk}}{i}}{\frac{1-\nu^{n}}{i}} = 1 + \nu^n + \nu^{2n} + \cdots + \nu^{n(k-1)} = \ddot{a}_{\overline{k}|(1+i)^n-1}.$$

$\square$

An application of $a_{\overline{nk}|i} = a_{\overline{n}|i}\ddot{a}_{\overline{k}|(1+i)^n-1}$.

Example 18

Carrie receives 200,000 from a life insurance policy. She uses the fund to purchase to two different annuities, each costing 100,000. The first annuity is a 24–year annuity–immediate paying k per year to herself. The second annuity is a 8 year annuity–immediate paying 2k per year to her boyfriend. Both annuities are based upon an annual effective interest rate of i, $i > 0$. Determine i.

An application of $a_{\overline{nk}|i} = a_{\overline{n}|i}\ddot{a}_{\overline{k}|(1+i)^n-1}$.

Example 18

Carrie receives 200,000 from a life insurance policy. She uses the fund to purchase to two different annuities, each costing 100,000. The first annuity is a 24–year annuity–immediate paying k per year to herself. The second annuity is a 8 year annuity–immediate paying $2k$ per year to her boyfriend. Both annuities are based upon an annual effective interest rate of i, $i > 0$. Determine i.

Solution: We have that $100000 = ka_{\overline{24}|i} = 2ka_{\overline{8}|i}$. So, $2 = \frac{a_{\overline{24}|i}}{a_{\overline{8}|i}} = \ddot{a}_{\overline{3}|(1+i)^8-1}$. Using the calculator, we get that $(1+i)^8 - 1 = 0.618034$ and $i = 6.1997\%$.

Theorem 9

For integers $k, n \geq 1$, $\frac{a_{\overline{nk}|i}}{s_{\overline{n}|i}} = a_{\overline{k}|(1+i)^n-1}$.

Theorem 9

For integers $k, n \geq 1$, $\frac{a_{\overline{nk}|i}}{s_{\overline{n}|i}} = a_{\overline{k}|(1+i)^n-1}$.

Proof.

By the previous theorem, $a_{\overline{nk}|i} = a_{\overline{n}|i}\ddot{a}_{\overline{k}|(1+i)^n-1}$. The claim follows noticing that $a_{\overline{n}|i} = (1+i)^{-n}s_{\overline{n}|i}$ and $\ddot{a}_{\overline{k}|(1+i)^n-1} = (1+i)^n a_{\overline{k}|(1+i)^n-1}$. □

An application of $a_{\overline{nk}|i} = s_{\overline{n}|i} a_{\overline{k}|(1+i)^n-1}$.

Example 19

The present value of a $4n$*–year annuity–immediate of 1 at the end of every year is* 16.663*. The present value of a* $4n$*–year annuity–immediate of 1 at the end of every fourth year is* 3.924*. Find* n *and* i*.*

An application of $a_{\overline{nk}|i} = s_{\overline{n}|i} a_{\overline{k}|(1+i)^n-1}$.

Example 19

The present value of a $4n$*–year annuity–immediate of 1 at the end of every year is* 16.663*. The present value of a* $4n$*–year annuity–immediate of 1 at the end of every fourth year is* 3.924*. Find* n *and* i*.*

Solution: We know that $16.663 = a_{\overline{4n}|i}$ and $3.924 = a_{\overline{n}|(1+i)^4-1}$. Dividing the first equation over the second one, we get that

$$4.246432 = \frac{16.663}{3.924} = \frac{a_{\overline{4n}|i}}{a_{\overline{n}|(1+i)^4-1}} = s_{\overline{4}|i}$$

and $i = 4\%$. From the equation $16.663 = a_{\overline{4n}|4\%}$, we get $n = 7$.

Since $a_{\overline{n}|i} = \sum_{j=1}^{n}(1+i)^{-j}$ as a function of n, $a_{\overline{n}|i}$ increases with n. As a function of i, $i \geq 0$, $a_{\overline{n}|i}$ decreases with i. We have that $a_{\overline{n}|0} = n$ and $a_{\overline{n}|\infty} = 0$. So, $n > a_{\overline{n}|i} > 0$ for $i > 0$; and $a_{\overline{n}|i} > n$ for $0 > i > -1$.

It is proved in the manual that

$$a_{\overline{n}|i} = \sum_{j=1}^{\infty} \frac{(n+j-1)!}{(n-1)!j!}(-i)^{j-1}.$$

The first order Taylor expansion of $a_{\overline{n}|i}$ on i is

$$a_{\overline{n}|i} \approx n - \frac{n(n+1)}{2}i.$$

The second order Taylor expansion of $a_{\overline{n}|i}$ on i is

$$a_{\overline{n}|i} \approx n - \frac{n(n+1)}{2}i + \frac{n(n+1)(n+2)}{6}i^2.$$

