# Manual for SOA Exam FM/CAS Exam 2.

## Chapter 3. Annuities.

## Section 3.3. Level payment perpetuities.



Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2,
Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

A **perpetuity** is a series of payments made forever along equal intervals of time. By a change of units, we will assume that intervals have unit length.

The payments can be made either at the beginning or at the end of the intervals.

A perpetuity has level payments if all payments $C_j$, $j \geq 0$, are equal.

A perpetuity has non–level payments if some payments $C_j$ are different from other ones.



 

For a **perpetuity–immediate** the payments are made at the end of the periods of time, i.e. at the times $1, 2, \ldots$. So, a perpetuity–immediate is a cashflow of the type:

| Payments | $C_1$ | $C_2$ | $C_3$ | $\cdots$ |
|---|---|---|---|---|
| Time | 1 | 2 | 3 | $\cdots$ |

For a **perpetuity–due** the payments are made at the beginning of the intervals of time, i.e. at the times $0, 1, 2, \ldots$. So, a perpetuity immediate is a cashflow of the type:

| Payments | $C_0$ | $C_1$ | $C_2$ | $\cdots$ |
|---|---|---|---|---|
| Time | 0 | 1 | 2 | $\cdots$ |



 

## Theorem 1

*The cashflow value of a perpetuity–immediate with level payments of one is*

| *Payments* | 1 | 1 | 1 | $\cdots$ |
|---|---|---|---|---|
| *Time* | 1 | 2 | 3 | $\cdots$ |

*The present value of a perpetuity–immediate with level payments of one is*

$$a_{\overline{\infty}|i} = \nu + \nu^2 + \cdots = \frac{\nu}{1-\nu} = \frac{1}{i}.$$



 

## Example 1

*The present value of a series of payments of 3 at the end of every eight years, forever, is equal to 9.5. Calculate the effective annual rate of interest.*



 

## Example 1

*The present value of a series of payments of 3 at the end of every eight years, forever, is equal to 9.5. Calculate the effective annual rate of interest.*

**Solution:** The 8–year interest factor is $(1+i)^8$. So, the 8–year effective interest rate is $(1+i)^8 - 1$. We have that $9.5 = (3)a_{\overline{\infty}|(1+i)^8-1} = \frac{3}{(1+i)^8-1}$ and $i = \left(1 + \frac{3}{9.5}\right)^{1/8} - 1 = 3.489979511\%$.



 

## Theorem 2

*The cashflow value of a perpetuity–due with level payments of one is*

| *Payments* | 1 | 1 | 1 | $\cdots$ |
|---|---|---|---|---|
| *Time* | 0 | 1 | 2 | $\cdots$ |

*The present value of an perpetuity–due with level payments of one is*

$$\ddot{a}_{\overline{\infty}|i} = 1 + \nu + \nu^2 + \cdots = \frac{1}{1-\nu} = \frac{1}{d} = \frac{1+i}{i}.$$



 

## Example 2

*John uses his retirement fund to buy a perpetuity–due of 20,000 per year based on an annual nominal yield of interest* $i = 8\%$ *compounded monthly. Find John's retirement fund.*



 

## Example 2

*John uses his retirement fund to buy a perpetuity–due of 20,000 per year based on an annual nominal yield of interest* $i = 8\%$ *compounded monthly. Find John's retirement fund.*

**Solution:** *Since* $i^{(12)} = 8\%$, $i = 8.299950681\%$. *John's retirement fund is worth*

$$(20000)\ddot{a}_{\overline{\infty}|i} = (20000)\frac{1+i}{i} = (20000)\frac{1.08299950681}{0.08299950681} = 260963.8554.$$



 

## Example 3

*A perpetuity pays \$1 at the end of every year plus an additional \$1 at the end of every second year. The effective rate of interest is* $i = 5\%$*. Find the present value of the perpetuity at time* $0$*.*



 

**Solution:** The cashflow of the perpetuity is

| Payments | 1 | 2 | 1 | 2 | 1 | 2 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| Time | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ |

This cashflow can be decomposed into the cashflows:

| Payments | 1 | 1 | 1 | $\cdots$ |
|---|---|---|---|---|
| Time | 1 | 3 | 5 | $\cdots$ |

and

| Payments | 2 | 2 | 2 | $\cdots$ |
|---|---|---|---|---|
| Time | 2 | 4 | 6 | $\cdots$ |

The present value at time 0 of the first part of the cashflow is $\frac{1+i}{(1+i)^2-1}$. The present value at time 0 of the second part of the cashflow is $\frac{2}{(1+i)^2-1}$. Hence, the present value at time of the total cashflow is

$$\frac{1+i}{(1+i)^2-1}+\frac{2}{(1+i)^2-1}=\frac{3+i}{i(2+i)}=\frac{3+0.05}{0.05(2+0.05)}=29.7561.$$



 

## Theorem 3

*The cashflow value of the perpetuity–immediate with level payments of one per year and m payments per year is*

| *Payments* | 0 | $\frac{1}{m}$ | $\frac{1}{m}$ | $\cdots$ |
|---|---|---|---|---|
| *Time* | 0 | $\frac{1}{m}$ | $\frac{2}{m}$ | $\cdots$ |

*The present value of the previous perpetuity–immediate is*

$$a^{(m)}_{\overline{\infty}|i} = \frac{1}{m} \cdot \left(\nu^{1/m} + \nu^{2/m} + \cdots\right) = \frac{1}{m} \cdot \nu^{1/m} \cdot \frac{1}{1 - \nu^{1/m}}$$

$$= \frac{1}{m(\nu^{-1/m} - 1)} = \frac{1}{i^{(m)}}.$$



 

## Example 4

*A perpetuity pays $x$ at the end of each month. The nominal annual rate of interest compounded monthly is $i^{(12)}$. Calculate the percentage of increase in the value of this perpetuity if the nominal annual rate of interest compounded monthly decreases by 10%.*

## Example 4

*A perpetuity pays $x$ at the end of each month. The nominal annual rate of interest compounded monthly is $i^{(12)}$. Calculate the percentage of increase in the value of this perpetuity if the nominal annual rate of interest compounded monthly decreases by 10%.*

**Solution:** At the rate $i^{(12)}$, the present value of the perpetuity is $\frac{x}{i^{(12)}}$. At the rate $i^{(12)}(0.9)$, the present value of the perpetuity is $\frac{x}{i^{(12)}(0.9)}$. The percentage of increase in the value of this perpetuity is

$$\frac{\frac{x}{i^{(12)}(0.9)} - \frac{x}{i^{(12)}}}{\frac{x}{i^{(12)}}} = \frac{1}{0.9} - 1 = 11.11111111\%.$$



 

## Theorem 4

*The cashflow value of the perpetuity–due with level payments of one per year and m payments per year is*

| Payments | $\frac{1}{m}$ | $\frac{1}{m}$ | $\frac{1}{m}$ | $\cdots$ |
|---|---|---|---|---|
| Time | $0$ | $\frac{1}{m}$ | $\frac{2}{m}$ | $\cdots$ |

*The present value of an perpetuity–due with level payments of one per year and m payments per year is*

$$\ddot{a}^{(m)}_{\overline{\infty}|i} = \frac{1}{m} \cdot \left(1 + \nu^{1/m} + \nu^{2/m} + \cdots\right) = \frac{1}{m} \cdot \frac{1}{1 - \nu^{1/m}} = \frac{1}{d^{(m)}}.$$



 

## Example 5

*The present value of a series of payments of \$500 at the beginning of every month years, forever, is equal to \$10000. Calculate the nominal annual discount rate compounded monthly.*

**Solution:** We have that

$$10000 = 500\frac{1}{d^{(12)}}.$$

and $d^{(12)} = \frac{500}{10000} = 0.05$.



 