Manual for SOA Exam FM/CAS Exam 2.

Chapter 3. Annuities.

Section 3.4. Non-level payment annuities and perpetuities.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2,
Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Theorem 1

(**Geometric Annuity**) *Let $i, r > -1$. The present value of the annuity*

Payments	1	$1+r$	$(1+r)^2$	$\cdots$	$(1+r)^{n-1}$
Time	1	2	3	$\cdots$	n

is

$$(Ga)_{\overline{n}|i,r} = \frac{1}{1+r} a_{\overline{n}|\frac{i-r}{1+r}} = \frac{1}{1+i} \ddot{a}_{\overline{n}|\frac{i-r}{1+r}}.$$

Proof: Let $i' = \frac{i-r}{1+r}$. Then, $\frac{1+i}{1+r} = 1 + i'$. The present value of the cashflow at time 0 is

$$\sum_{j=1}^{n}(1+r)^{j-1}(1+i)^{-j} = \frac{1}{1+r}\sum_{j=1}^{n}\left(\frac{1+i}{1+r}\right)^{-j}$$
$$=\frac{1}{1+r}\sum_{j=1}^{n}\left(1+i'\right)^{-j} = \frac{1}{1+r}a_{\overline{n}|i'}.$$

Using that $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$, we get that

$$\frac{1}{1+r}a_{\overline{n}|i'} = \frac{1}{1+r}\frac{1}{1+i'}\ddot{a}_{\overline{n}|i'} = \frac{1}{1+i}\ddot{a}_{\overline{n}|i'}.$$

Example 1

An annuity provides for 10 annuals payments, the first payment a year hence being \$2600. The payments increase in such a way that each payment is 3% *greater than the previous one. The annual effective rate of interest is* 4%. *Find the present value of this annuity.*

Example 1

An annuity provides for 10 annuals payments, the first payment a year hence being $2600. The payments increase in such a way that each payment is 3% greater than the previous one. The annual effective rate of interest is 4%. Find the present value of this annuity.

Solution: The cashflow is

Payments	2600	2600(1.03)	$2600(1.03)^2$	$\cdots$	$2600(1.03)^9$
Time	1	2	3	$\cdots$	10

The present value at time 0 of the annuity is

$$(2600)\,(Ga)_{\overline{n}|i,r} = \frac{2600}{1.03} a_{\overline{10}|\frac{0.04-0.03}{1.03}} = 2524.271845 a_{\overline{10}|0.9708737864\%}$$

$=23945.54454.$

Example 2

An annuity provides for 20 annuals payments, the first payment a year hence being \$4500. The payments increase in such a way that each payment is 4.5% *greater than the previous one. The annual effective rate of interest is* 4.5%*. Find the present value of this annuity.*

Example 2

An annuity provides for 20 annuals payments, the first payment a year hence being \$4500. The payments increase in such a way that each payment is 4.5% *greater than the previous one. The annual effective rate of interest is* 4.5%. *Find the present value of this annuity.*

Solution: The cashflow is

Payments	4500	4500(1.045)	$4500(1.045)^2$	$\cdots$	$4500(1.045)^{19}$
Time	1	2	3	$\cdots$	20

The present value at time 0 of the annuity is

$$(4500)\,(Ga)_{\overline{n}|i,r} = (4500)\frac{1}{1.045}a_{\overline{20}|\frac{0.045-0.045}{1.045}} = 4500\frac{20}{1.045} = 86124.40191.$$

Example 3

Chris makes annual deposits into a bank account at the beginning of each year for 10 years. Chris initial deposit is equal to 100, with each subsequent deposit k% greater than the previous year deposit. The bank credits interest at an annual effective rate of 4.5%. At the end of 10 years, the accumulated amount in Chris account is equal to 1657.22. Calculate k.

Example 3

Chris makes annual deposits into a bank account at the beginning of each year for 10 years. Chris initial deposit is equal to 100, with each subsequent deposit k% greater than the previous year deposit. The bank credits interest at an annual effective rate of 4.5%. At the end of 10 years, the accumulated amount in Chris account is equal to 1657.22. Calculate k.

Solution: The cashflow is

Payments	100	$100(1+r)$	$100(1+r)^2$	$\cdots$	$100(1+r)^9$
Time	0	1	2	$\cdots$	9

The accumulated amount at the end of 10 years is
$1657.22 = (100)\frac{1}{1+i}\ddot{a}_{\overline{n}|\frac{i-r}{1+r}}(1+i)^{11}$. Hence,
$\ddot{a}_{\overline{10}|\frac{0.045-r}{1+r}} = \frac{(1657.22)(1.045)^{-10}}{100} = 10.67129833$,
$\frac{0.045-r}{1+r} = -0.0141511755$, $r = \frac{0.045+0.0141511755}{1-0.0141511755} = 6\%$ and $k = 6$.

Corollary 1

The present value of the perpetuity

Payments	1	$1+r$	$(1+r)^2$	$\cdots$	$(1+r)^{n-1}$	$\cdots$
Time	1	2	3	$\cdots$	n	$\cdots$

is

$$(Ga)_{\overline{\infty}|i,r} = \begin{cases} \frac{1}{i-r} & \text{if } i > r, \\ \infty & \text{if } i \leq r. \end{cases}$$

Corollary 1

The present value of the perpetuity

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$$(Ga)_{\overline{\infty}|i,r} = \begin{cases} \frac{1}{i-r} & \text{if } i > r, \\ \infty & \text{if } i \le r. \end{cases}$$

Proof.

If $i > r$, $(Ga)_{\overline{\infty}|i,r} = \frac{1}{1+r} a_{\overline{\infty}|\frac{i-r}{1+r}} = \frac{1}{1+r}\frac{1}{\frac{i-r}{1+r}} = \frac{1}{i-r}$.

If $i \le r$, $(Ga)_{\overline{\infty}|i,r} = \frac{1}{1+r} a_{\overline{\infty}|\frac{i-r}{1+r}} = \infty$. $\square$

Example 4

An perpetuity–immediate provides annual payments. The first payment of 13000 is one year from now. Each subsequent payment is 3.5% more than the one preceding it. The annual effective rate of interest is $i = 6\%$. *Find the present value of this perpetuity.*

Example 4

An perpetuity–immediate provides annual payments. The first payment of 13000 is one year from now. Each subsequent payment is 3.5% more than the one preceding it. The annual effective rate of interest is $i = 6\%$. Find the present value of this perpetuity.

Solution: The present value is

$$(13000)\,(Ga)_{\overline{\infty}|i,r} = (13000)\frac{1}{i-r} = (13000)\frac{1}{0.06-0.035} = 520000.$$

Theorem 2

(**Increasing Annuity**) *The present value of the annuity*

Payments	1	2	3	$\cdots$	n
Time	1	2	3	$\cdots$	n

is

$$(Ia)_{\overline{n}|i} = \frac{a_{\overline{n}|i}(1+i) - n\nu^n}{i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i}.$$

The accumulated value of this cashflow at time n is

$$(Is)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i}.$$

Proof:

The cashflow is the sum of the cashflows:

cashflow 1	1	1	1	$\cdots$	1	1
cashflow 2	0	1	1	$\cdots$	1	1
cashflow 3	0	0	1	$\cdots$	1	1
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
cashflow n	0	0	0	$\cdots$	0	1
Time	1	2	3	$\cdots$	$n-1$	n

The future value at time n of all these cashflows is

$$(Is)_{\overline{n}|i} = \sum_{j=1}^{n} s_{\overline{j}|i} = \sum_{j=1}^{n} \frac{(1+i)^j - 1}{i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i}.$$

In the calculator, it is possible to find $\ddot{a}_{\overline{n}|i} - n\nu^n$ in one computation. With payments set–up at the beginning, we enter n in $\boxed{\text{N}}$, i in $\boxed{\text{I/Y}}$, 1 in $\boxed{\text{PMT}}$ and $-n$ in $\boxed{\text{FV}}$. We ask the calculator to compute $\boxed{\text{PV}}$.

Example 5

Find the present value at time 0 of an annuity–immediate such that the payments start at 1, each payment thereafter increases by 1 until reaching 10, and they remain at that level until 25 *payments in total are made. The effective annual rate of interest is* 4%.

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Find the present value at time 0 of an annuity–immediate such that the payments start at 1, each payment thereafter increases by 1 until reaching 10, and they remain at that level until 25 payments in total are made. The effective annual rate of interest is 4%.

Solution: The cashflow is

Payments	1	2	3	$\cdots$	10	10	$\cdots$	10
Time	1	2	3	$\cdots$	10	11	$\cdots$	25

The present value at time 0 of the perpetuity is

$$\begin{aligned}&(Ia)_{\overline{10}|0.04} + (1+0.04)^{-10} 10 a_{\overline{15}|0.04}\\ =&\frac{\ddot{a}_{\overline{n}|4\%} - 10(1+0.04)^{-10}}{0.04} + (1+0.04)^{-10}(10)a_{\overline{15}|0.04}\\ =&41.99224806 + 75.11184164 = 117.1040897.\end{aligned}$$

Theorem 3

(**Decreasing Annuity**) *The present value of the annuity*

Payments	n	$n-1$	$n-2$	$\cdots$	1
Time	1	2	3	$\cdots$	n

is

$$(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i}.$$

The accumulated value of this cashflow at time n is

$$(Ds)_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i}.$$

Proof.

The cashflow is the sum of the cashflows:

cashflow 1	1	1	1	$\cdots$	1	1
cashflow 2	1	1	1	$\cdots$	1	0
cashflow 3	1	1	1	$\cdots$	0	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
cashflow n	1	0	0	$\cdots$	0	0
Time	1	2	3	$\cdots$	$n-1$	n

The present value at time 0 of all these cashflows is

$$\sum_{j=1}^{n} a_{\overline{j}|i} = \sum_{j=1}^{n} \frac{1-(1+i)^{-j}}{i} = \frac{n - a_{\overline{n}|i}}{i}.$$

$\square$

In the calculator, it is possible to find $n(1+i)^n - s_{\overline{n}|i}$ in one computation. We enter n in $\boxed{\text{N}}$, i in $\boxed{\text{I/Y}}$, 1 in $\boxed{\text{PMT}}$ and $-n$ in $\boxed{\text{PV}}$. We ask the calculator to compute $\boxed{\text{FV}}$.

Example 6

Find the present value of a 15–year decreasing annuity–immediate paying 150000 *the first year and decreasing by* 10000 *each year thereafter. The effective annual interest rate of 4.5%.*

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Find the present value of a 15–year decreasing annuity–immediate paying 150000 *the first year and decreasing by* 10000 *each year thereafter. The effective annual interest rate of 4.5%.*

Solution: The cashflow of payments is

Payments	(15)(10000)	(14)(10000)	$\cdots$	(1)(10000)
Time	1	2	$\cdots$	15

The present value of this cashflow is

$$(10000)\,(Da)_{\overline{15}|4.5\%} = (10000)\frac{15 - a_{\overline{15}|4.5\%}}{0.045} = 946767.616.$$

Theorem 4

(**Increasing Perpetuity**) *The present value of the perpetuity*

Payments	1	2	3	$\cdots$
Time	1	2	3	$\cdots$

is $(Ia)_{\overline{\infty}|i} = \frac{1+i}{i^2}$.

Proof:

The cashflow is the sum of the infinitely many cashflows:

cashflow 1	1	1	1	$\cdots$	1	1	$\cdots$	$\cdots$
cashflow 2	0	1	1	$\cdots$	1	1	$\cdots$	$\cdots$
cashflow 3	0	0	1	$\cdots$	1	1	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Time	1	2	3	$\cdots$	$n-1$	n	$\cdots$	$\cdots$

The present value at time 0 of all these cashflows is

$$\sum_{j=1}^{\infty} \frac{1}{i}(1+i)^{-(j-1)} = \frac{1}{i}\frac{1}{1-\frac{1}{1+i}} = \frac{1+i}{i^2}.$$

Example 7

An investor is considering the purchase of 500 ordinary shares in a company. This company pays dividends at the end of each year. The next payment is one year from now and it is \$3 per share. The investor believes that each subsequent payment per share will increase by \$1 each year forever. Calculate the present value of this dividend stream at a rate of interest of 6.5% per annum effective.

Solution: The cashflow of payments is

Payments	3	4	5	$\cdots$
Time	1	2	3	$\cdots$

The cashflow is the sum of the following cashflows

Payments	2	2	2	$\cdots$
Payments	1	2	3	$\cdots$
Time	1	2	3	$\cdots$

Hence, the present value of this dividend stream is

$$\begin{aligned}&(500)(2)a_{\overline{\infty}|6.5\%} + (500)(1)\,(Ia)_{\overline{\infty}|6.5\%}\\ =&(500)(2)\frac{1}{0.065} + (500)(1)\frac{1.065}{0.065^2}\\ =&15384.6154 + 126035.503 = 141420.118.\end{aligned}$$

Theorem 5

(**Rainbow Immediate**) *The present value of the annuity*

Payments	1	2	$\cdots$	$n-1$	n	$n-1$	$\cdots$	2	1
Time	1	2	$\cdots$	$n-1$	n	$n+1$	$\cdots$	$2n-2$	$2n-1$

is $\ddot{a}_{\overline{n}|i} a_{\overline{n}|i}$.

Proof: The value of the annuity is

$$
\begin{aligned}
& (Ia)_{\overline{n}|i} + \nu^n (Da)_{\overline{n-1}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i} + \frac{\nu^n(n-1-a_{\overline{n-1}|i})}{i} = \frac{\ddot{a}_{\overline{n}|i} - \nu^n(1+a_{\overline{n-1}|i})}{i} \\
= \; & \frac{(1+i)a_{\overline{n}|i} - \nu^n(1+i)a_{\overline{n}|i}}{i} = \frac{(1+i)a_{\overline{n}|i}(1-\nu^n)}{i} = (1+i)a_{\overline{n}|i}a_{\overline{n}|i} = \ddot{a}_{\overline{n}|i}a_{\overline{n}|i}
\end{aligned}
$$

Example 8

A 15 year annuity pays 1000 at the end of year 1 and increases by 1000 each year until the payment is 8000 at the end of year 8. Payments then decrease by 1000 each year until a payment of 1000 is paid at the end of year 15. The annual effective interest rate of 6.5%. Compute the present value of this annuity.

Example 8

A 15 year annuity pays 1000 at the end of year 1 and increases by 1000 each year until the payment is 8000 at the end of year 8. Payments then decrease by 1000 each year until a payment of 1000 is paid at the end of year 15. The annual effective interest rate of 6.5%. Compute the present value of this annuity.

Solution: The cashflow is

Paym.	1000	2000	$\cdots$	7000	8000	7000	$\cdots$	2000	1000
Time	1	2	$\cdots$	7	8	9	$\cdots$	14	15

The present value is

$$1000\ddot{a}_{\overline{8}|i}a_{\overline{8}|i} = (1000)(6.08875096)(6.48451977) = 39482.626.$$

Theorem 6

(**Paused Rainbow Immediate**) *The present value of the annuity*

Paym.	1	2	$\cdots$	$n-1$	n	n	$n-1$	$\cdots$	2	1
Time	1	2	$\cdots$	$n-1$	n	$n+1$	$n+2$	$\cdots$	$2n-1$	$2n$

is $\ddot{a}_{\overline{n+1}|i} a_{\overline{n+1}|i}$.

Theorem 6

(**Paused Rainbow Immediate**) *The present value of the annuity*

Paym.	1	2	$\cdots$	$n-1$	n	n	$n-1$	$\cdots$	2	1
Time	1	2	$\cdots$	$n-1$	n	$n+1$	$n+2$	$\cdots$	$2n-1$	$2n$

is $\ddot{a}_{\overline{n+1}|i} a_{\overline{n+1}|i}$.

Proof: The present value of the annuity is

$$
\begin{aligned}
& (Ia)_{\overline{n}|i} + \nu^n (Da)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i} + \frac{\nu^n(n - a_{\overline{n}|i})}{i} = \frac{\ddot{a}_{\overline{n}|i} - \nu^n a_{\overline{n}|i}}{i} \\
= \; & \frac{(1+i)a_{\overline{n}|i} - (1 - ia_{\overline{n}|i})a_{\overline{n}|i}}{i} \\
= \; & a_{\overline{n}|i}(1 + a_{\overline{n}|i}) = a_{\overline{n}|i}\ddot{a}_{\overline{n+1}|i}.
\end{aligned}
$$

Example 9

A 20 year annuity pays 5000 at the end of year 1 and increases by 5000 each year until the payment is 50000 at the end of year 10. The payment remains constant for one year. Payments then decrease by 5000 each year until a payment of 5000 is paid at the end of year 20. The annual effective interest rate of 4%. Compute the present value of this annuity.

Example 9

A 20 year annuity pays 5000 at the end of year 1 and increases by 5000 each year until the payment is 50000 at the end of year 10. The payment remains constant for one year. Payments then decrease by 5000 each year until a payment of 5000 is paid at the end of year 20. The annual effective interest rate of 4%. Compute the present value of this annuity.

Solution: The cashflow is

Paym.	(1)(5000)	$\cdots$	(10)5000	(10)5000	$\cdots$	(1)(5000)
Time	1	$\cdots$	10	11	$\cdots$	20

The present value is

$$5000\ddot{a}_{\overline{11}|i}a_{\overline{10}|i} = (5000)(8.11089578)(9.11089578) = 369487.631$$

Theorem 7

The present value of the annuity

Payments	0	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{2}{m}$	$\cdots$	$\cdots$	$\frac{n}{m}$
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	1	$1+\frac{1}{m}$	$\cdots$	2	$\cdots$	$\cdots$	n

is

$$(Ia)^{(m)}_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i^{(m)}}.$$

For the proof, see the manual.

Theorem 8

The present value of the annuity

Payments	0	$\frac{1}{m^2}$	$\frac{2}{m^2}$	$\frac{3}{m^2}$	$\cdots$	$\cdots$	$\frac{n}{m^2}$
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	$\frac{3}{m}$	$\cdots$	$\cdots$	n

is

$$\left(I^{(m)}a\right)^{(m)}_{\overline{n}|i} = \frac{\ddot{a}^{(m)}_{\overline{n}|i} - n\nu^n}{i^{(m)}}.$$

For the proof, see the manual.

