# Manual for SOA Exam FM/CAS Exam 2.

## Chapter 4. Amortization and sinking bonds.
## Section 4.1. Amortization schedules.



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In this chapter, we study different problems related with the payment of a loan. Suppose that a **borrower** (also called **debtor**) takes a loan from a **lender**. The borrower will make payments which eventually will repay the loan. Payments made by the borrower can be applied to the outstanding balance or not. According with the **amortization method**, all the payments made by the borrower reduce the outstanding balance of the loan. When a loan is paid usually, the total amount of loan payments exceed the loan amount. The **finance charge** is the total amount of interest paid (the total payments minus the loan payments).



 

The simplest way to pay a loan is by unique payment. Suppose that a borrower takes a loan with amount $L$ at time zero and the lender charges an annual effective rate of interest of $i$. If the borrower pays the loan with a lump sum $P$ at time $n$, then $P = L(1+i)^n$. The finance charge in this situation is $L(1+i)^n - L$.



 

## Example 1

*Juan borrows \$35,000 for four years at an annual nominal interest rate of 7.5% convertible monthly. Juan will pay the loan with a unique payment at the end of four years.*
*(i) Find the amount of this payment.*
*(ii) Find the finance charge which Juan is charged in this loan.*



 

## Example 1

*Juan borrows \$35,000 for four years at an annual nominal interest rate of 7.5% convertible monthly. Juan will pay the loan with a unique payment at the end of four years.*
*(i) Find the amount of this payment.*
*(ii) Find the finance charge which Juan is charged in this loan.*

**Solution:** (i) The amount of the loan payment is
$(35000)\left(1+\frac{0.075}{12}\right)^{(12)(4)} = 47200.97.$
(ii) The finance charge which Juan is charged in this loan is
$47200.97 - 35000 = 12200.97.$

Suppose that a borrower takes a loan of $L$ at time 0 and repays the loan in a series of payments $C_1, \dots, C_n$ at times $t_1, \dots, t_n$, where $0 < t_1 < t_2 < \cdots < t_n$. The debtor cashflow is

| Inflows | $L$ | $-C_1$ | $-C_2$ | $-C_n$ | $\cdots$ | $-C_n$ |
|---|---|---|---|---|---|---|
| Time | 0 | $t_1$ | $t_2$ | $t_3$ | $\cdots$ | $t_n$ |

Assume that the loan increases with a certain accumulation function $a(t)$, $t \geq 0$. Since the loan will be repaid, the present value a time zero (or any other time) of this cashflow is zero. Hence

$$L = \sum_{j=1}^{n} \frac{C_j}{a(t_j)}.$$

The finance charge for this loan is

$$\sum_{j=1}^{n} C_j - L = \sum_{j=1}^{n} C_j - \sum_{j=1}^{n} \frac{C_j}{a(t_j)} = \sum_{j=1}^{n} C_j \left(1 - \frac{1}{a(t_j)}\right).$$



 

According with the **retrospective method**, the outstanding balance at certain point is the present value of the loan at that time minus the present value of the payments made at that time. For the cashflow

| Inflows | $L$ | $-C_1$ | $-C_2$ | $-C_n$ | $\cdots$ | $-C_n$ |
|---|---|---|---|---|---|---|
| Time | 0 | $t_1$ | $t_2$ | $t_3$ | $\cdots$ | $t_n$ |

the outstanding balance immediately after the $k$–th payment, is

$$B_k = La(t_k) - \sum_{j=1}^{k} \frac{a(t_k)C_j}{a(t_j)}.$$

Of course, we have that $B_0 = L$, $B_n = 0$.



 

According to the **prospective method**, the outstanding balance after the $k$–th payment is equal to the present value of the remaining payments.
For the cashflow

| Inflows | $L$ | $-C_1$ | $-C_2$ | $-C_n$ | $\cdots$ | $-C_n$ |
|---|---|---|---|---|---|---|
| Time | 0 | $t_1$ | $t_2$ | $t_3$ | $\cdots$ | $t_n$ |

the outstanding balance immediately after the $k$–th payment, is

$$B_k = \sum_{j=k+1}^{n} \frac{a(t_k)C_j}{a(t_j)}$$

Of course, we have that

$$La(t_k) - \sum_{j=1}^{k} \frac{a(t_k)C_j}{a(t_j)} = \sum_{j=k+1}^{n} \frac{a(t_k)C_j}{a(t_j)}.$$


 

An **inductive relation for the outstanding balance** is

$$B_k = B_{k-1}\frac{a(t_k)}{a(t_{k-1})} - C_k.$$

Previous relation says that the outstanding balance after the $k$–th payment is the accumulation of the previous outstanding balance minus the amount of the payment made.



 

During the period $[t_{k-1}, t_k]$, the **amount of interest accrued** is

$$I_k = B_{k-1}\left(\frac{a(t_k)}{a(t_{k-1})} - 1\right).$$

Immediately before the $k$–th payment, the outstanding balance is $B_{k-1} + I_k = B_{k-1}\frac{a(t_k)}{a(t_{k-1})}$. Immediately after the $k$–th payment, the outstanding balance is $B_k = B_{k-1} + I_k - C_k$. The $k$–th payment $C_k$ can be split as $I_k$ plus $C_k - I_k$. $I_k$ is called the **interest portion** of the $k$–th payment. $C_k - I_k$ is called the **principal portion** of the $k$–the payment. If $C_k - I_k < 0$, then the outstanding balance increases during the $k$–th period. Notice that

$$C_k - I_k = C_k - B_{k-1}\left(\frac{a(t_k)}{a(t_{k-1})} - 1\right) = B_{k-1} - B_k$$

is the reduction on principal made during the the $k$–period. The total amount of reduction on principal is equal to the loan amount: $\sum_{k=1}^{n}(B_k - B_{k-1}) = B_n - B_0 = L$.



 

Under compound interest,

$$L = \sum_{j=1}^{n} C_j (1+i)^{-t_j}.$$

The outstanding balance immediately after the $k$–th payment is

$$B_k = L(1+i)^{t_k} - \sum_{j=1}^{k} C_j (1+i)^{t_k - t_j} = \sum_{j=k+1}^{n} C_j (1+i)^{t_k - t_j}.$$

The inductive relation for outstanding balances is

$$B_k = B_{k-1}(1+i)^{t_k - t_{k-1}} - C_k.$$

The amount of interest accrued during the period $[t_{k-1}, t_k]$ is

$$I_k = B_{k-1}\left((1+i)^{t_k - t_{k-1}} - 1\right).$$

The principal portion of the $k$–the payment is

$$C_k - I_k = C_k - B_{k-1}\left((1+i)^{t_k - t_{k-1}} - 1\right) = B_{k-1} - B_k.$$

The finance charge is $\sum_{j=1}^{n} C_j - L = \sum_{j=1}^{n} C_j \left(1 - (1+i)^{-t_j}\right)$.

Usually, we consider payments made at equally spaced intervals of time and compound interest. Suppose that a borrower takes a loan $L$ at time 0 and repays the loan in a series of level payments $C_1, \dots, C_n$ at times $t_0, 2t_0, \dots, nt_0$. By a change of units, we may assume that $t_0 = 1$. Hence, the debtor cashflow is

| Inflows | $L$ | $-C_1$ | $-C_2$ | $-C_n$ | $\cdots$ | $-C_n$ |
|---|---|---|---|---|---|---|
| Time | 0 | 1 | 2 | 3 | $\cdots$ | $n$ |

Let $i$ be the effective rate of interest per period. Then, we have that

$$L = \sum_{j=1}^{n} C_j (1+i)^{-j}.$$



 

The outstanding balance immediately after the $k$–th payment, is

$$B_k = L(1+i)^k - \sum_{j=1}^{k} C_j (1+i)^{k-j} = \sum_{j=k+1}^{n} C_j (1+i)^{k-j}.$$

The amount of interest accrued during the $k$–th year is $iB_{k-1}$. The principal portion of the $k$–th payment is $C_k - iB_{k-1} = B_k - B_{k-1}$. Hence, the outstanding balance after the $k$–th payment is

$$B_k = B_{k-1} - (C_k - iB_{k-1}) = (1+i)B_{k-1} - C_k.$$

The finance charge is

$$\sum_{j=1}^{n} C_j - L = \sum_{j=1}^{n} C_j (1 - (1+i)^{-j}) = \sum_{j=1}^{n} C_j (1 - \nu^j).$$



 

## Example 2

*Roger buys a car for \$25,000 by making level payments at the end of the month for three years. Roger is charged an annual nominal interest rate of 8.5% compounded monthly in his loan.*
*(i) Find the amount of each monthly payment.*
*(ii) Find the total amount of payments made by Roger.*
*(iii) Find the total interest paid by Roger during the duration of the loan.*
*(iv) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the retrospective method.*
*(v) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the prospective method.*



 

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*(v) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the prospective method.*

**Solution:**

(i) Let $P$ be the monthly payment. We have that $25000 = Pa_{\overline{36}|0.085/12}$ and $P = 789.1884356$.



 

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*(iv) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the retrospective method.*
*(v) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the prospective method.*

**Solution:**

(ii) The total amount of payments made by Roger is $(36)(789.1884356) = 28410.78368$.


 

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*(iii) Find the total interest paid by Roger during the duration of the loan.*
*(iv) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the retrospective method.*
*(v) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the prospective method.*

**Solution:**

(iii) The total interest paid by Roger during the duration of the loan is $28410.78368 - 25000 = 3410.78368$.



 

## Example 2

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*(ii) Find the total amount of payments made by Roger.*
*(iii) Find the total interest paid by Roger during the duration of the loan.*
*(iv) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the retrospective method.*
*(v) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the prospective method.*

**Solution:**

(iv) The outstanding loan balance immediately after the 12–th payment has been made using the retrospective method. is

$$(25000)(1+0.085/12)^{12}-(789.1884356)s_{\overline{12}|0.085/12}=17361.71419.$$



 

## Example 2

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*(i) Find the amount of each monthly payment.*
*(ii) Find the total amount of payments made by Roger.*
*(iii) Find the total interest paid by Roger during the duration of the loan.*
*(iv) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the retrospective method.*
*(v) Calculate the outstanding loan balance immediately after the 12–th payment has been made using the prospective method.*

**Solution:**

(v) The outstanding loan balance immediately after the 12–th payment has been made using the prospective method is

$$(789.1884356)a_{\overline{24}|0.085/12} = 17361.71419.$$



 

## Example 3

*A loan is being repaid with 10 payments of \$3000 followed by 20 payments of \$5000 at the end of each year. The effective annual rate of interest is 4.5%.*
*(i) Calculate the amount of the loan.*
*(ii) Calculate the outstanding loan balance immediately after the 15–th payment has been made by both the prospective and the retrospective method.*
*(iii) Calculate the amounts of interest and principal paid in the 16–th payment.*



 

## Example 3

*A loan is being repaid with 10 payments of \$3000 followed by 20 payments of \$5000 at the end of each year. The effective annual rate of interest is 4.5%.*
*(i) Calculate the amount of the loan.*
*(ii) Calculate the outstanding loan balance immediately after the 15–th payment has been made by both the prospective and the retrospective method.*
*(iii) Calculate the amounts of interest and principal paid in the 16–th payment.*

**Solution:**

(i) The cashflow of payments is

| Inflows | 3000 | 3000 | ⋯ | 3000 | 5000 | 5000 | ⋯ | 5000 |
|---|---|---|---|---|---|---|---|---|
| Time | 1 | 2 | ⋯ | 10 | 11 | 12 | ⋯ | 30 |

The loan amount is $3000a_{\overline{10}|4.5\%} + (1.045)^{-10}5000a_{\overline{20}|4.5\%} = 23738.1545 + 41880.8518 = 65619.0063$.


 

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*(i) Calculate the amount of the loan.*
*(ii) Calculate the outstanding loan balance immediately after the 15–th payment has been made by both the prospective and the retrospective method.*
*(iii) Calculate the amounts of interest and principal paid in the 16–th payment.*

**Solution:**

(ii) The outstanding loan balance immediately after the 15–th payment using the retrospective method is

$$\begin{aligned} & 65619.0063(1.045)^{15} - 3000(1.045)^{5} s_{\overline{10}|4.5\%} - 5000 s_{\overline{5}|4.5\%} \\ = \ & 126991.311 - 45940.0337 - 27353.5486 = 53697.7287. \end{aligned}$$



 

## Example 3

*A loan is being repaid with 10 payments of \$3000 followed by 20 payments of \$5000 at the end of each year. The effective annual rate of interest is 4.5%.*
*(i) Calculate the amount of the loan.*
*(ii) Calculate the outstanding loan balance immediately after the 15–th payment has been made by both the prospective and the retrospective method.*
*(iii) Calculate the amounts of interest and principal paid in the 16–th payment.*

**Solution:**

The outstanding loan balance immediately after the 15–th payment using the prospective method is $5000a_{\overline{15}|4.5\%} = 53697.7286$.


 

## Example 3

*A loan is being repaid with 10 payments of \$3000 followed by 20 payments of \$5000 at the end of each year. The effective annual rate of interest is 4.5%.*
*(i) Calculate the amount of the loan.*
*(ii) Calculate the outstanding loan balance immediately after the 15–th payment has been made by both the prospective and the retrospective method.*
*(iii) Calculate the amounts of interest and principal paid in the 16–th payment.*

**Solution:**

(iii) The amount of interest paid in the 16–th payment is $(53697.7286)(0.045) = 2416.39779$. The amount of interest paid in the 16–th payment is $5000 - 2416.39779 = 2583.60221$.



 

Next, we consider the amortization method of repaying a loan with level payments made at the end of periods of the same length. Let $L$ be the amount borrowed. Let $P$ be the level payment. Let $n$ be the number of payments. Let $i$ be the effective rate of interest per payment period. The cashflow of payments is

| Inflows | $P$ | $P$ | $P$ | $\cdots$ | $P$ |
|---|---|---|---|---|---|
| Time | 1 | 2 | 3 | $\cdots$ | $n$ |

We have that $L = Pa_{\overline{n}|i}$.
The outstanding principal after the $k$–th payment is

$$B_k = L(1+i)^k - Ps_{\overline{k}|i} = P(a_{\overline{n}|i}(1+i)^k - s_{\overline{k}|i}) = Pa_{\overline{n-k}|i}.$$

The interest portion of the $k$–th payment is

$$iB_{k-1} = iPa_{\overline{n+1-k}|i} = P(1-\nu^{n+1-k}).$$

The principal reduction of the $k$–th payment is

$$B_{k-1} - B_k = P - iB_{k-1} = P\nu^{n+1-k}.$$



 

Using that $B_{k-1} = Pa_{\overline{n+1-k}|i}$ and $B_{k-1} - B_k = P\nu^{n+1-k}$, we get that $B_k = P(a_{\overline{n+1-k}|i} - \nu^{n+1-k})$. The outstanding principal after the $k$–th payment can be found using all these formulas

$$B_k = L(1+i)^k - Ps_{\overline{k}|i} = P(a_{\overline{n}|i}(1+i)^k - s_{\overline{k}|i})$$
$$= Pa_{\overline{n-k}|i} = P(a_{\overline{n+1-k}|i} - \nu^{n+1-k}).$$



 

The following is the amortization schedule for a loan of $L = Pa_{\overline{n|}i}$ with level payments of $P$.

| Period | Payment | Interest paid | Principal repaid | Outstanding balance |
|---|---|---|---|---|
| 0 | – | – | – | $L = Pa_{\overline{n\vert}i}$ |
| 1 | $P$ | $P(1-\nu^n)$ | $P\nu^n$ | $Pa_{\overline{n-1\vert}i}$ |
| 2 | $P$ | $P(1-\nu^{n-1})$ | $P\nu^{n-1}$ | $Pa_{\overline{n-2\vert}i}$ |
| 3 | $P$ | $P(1-\nu^{n-2})$ | $P\nu^{n-2}$ | $Pa_{\overline{n-3\vert}i}$ |
| ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... |
| $k$ | $P$ | $P(1-\nu^{n+1-k})$ | $P\nu^{n+1-k}$ | $Pa_{\overline{n-k\vert}i}$ |
| ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... |
| $n-1$ | $P$ | $P(1-\nu^2)$ | $P\nu^2$ | $Pa_{\overline{1\vert}i}$ |
| $n$ | $P$ | $P(1-\nu)$ | $P\nu$ | 0 |

## Example 4

*The following is the amortization schedule of a loan of $20,000.00 at an effective interest rate of 8% for 12 years.*

| Time | Payment amount | Interest paid | Principal reduction | Balance |
|---|---|---|---|---|
| 0 | — | — | — | 2000.00 |
| 1 | 2653.90 | 1600.00 | 1053.90 | 18946.10 |
| 2 | 2653.90 | 1515.69 | 1138.21 | 17807.89 |
| 3 | 2653.90 | 1424.63 | 1229.27 | 16578.62 |
| 4 | 2653.90 | 1326.29 | 1327.61 | 15251.01 |
| 5 | 2653.90 | 1220.08 | 1433.82 | 13817.19 |
| 6 | 2653.90 | 1105.38 | 1548.52 | 12268.67 |
| 7 | 2653.90 | 981.49 | 1672.41 | 10596.26 |
| 8 | 2653.90 | 847.70 | 1806.20 | 8790.06 |
| 9 | 2653.90 | 703.20 | 1950.70 | 6839.36 |
| 10 | 2653.90 | 547.15 | 2106.75 | 4732.61 |
| 11 | 2653.90 | 378.61 | 2275.29 | 2457.32 |
| 12 | 2653.91 | 196.59 | 2457.32 | 0.00 |



 

## Example 5

*A loan of 100,000 is being repaid by 15 equal annual installments made at the end of each year at 6% interest effective annually.*
*(i) Find the amount of each annual installment.*
*(ii) Find the finance charge of this loan.*
*(iii) Find how much interest is accrued in the first year.*
*(iv) Find how principal is repaid in the first payment.*
*(v) Find the balance in the loan immediately after the first payment.*



 

## Example 5

*A loan of 100,000 is being repaid by 15 equal annual installments made at the end of each year at 6% interest effective annually.*
*(i) Find the amount of each annual installment.*
*(ii) Find the finance charge of this loan.*
*(iii) Find how much interest is accrued in the first year.*
*(iv) Find how principal is repaid in the first payment.*
*(v) Find the balance in the loan immediately after the first payment.*

**Solution:**

(i)We solve $100000 = Pa_{\overline{15}|6\%}$ and get $P = 10296.2764$.



 

## Example 5

*A loan of 100,000 is being repaid by 15 equal annual installments made at the end of each year at 6% interest effective annually.*
*(i) Find the amount of each annual installment.*
*(ii) Find the finance charge of this loan.*
*(iii) Find how much interest is accrued in the first year.*
*(iv) Find how principal is repaid in the first payment.*
*(v) Find the balance in the loan immediately after the first payment.*

**Solution:**

(ii) The finance charge is $(15)(10296.2764) - 100000 = 54444.146$.



 

## Example 5

*A loan of 100,000 is being repaid by 15 equal annual installments made at the end of each year at 6% interest effective annually.*
*(i) Find the amount of each annual installment.*
*(ii) Find the finance charge of this loan.*
*(iii) Find how much interest is accrued in the first year.*
*(iv) Find how principal is repaid in the first payment.*
*(v) Find the balance in the loan immediately after the first payment.*

**Solution:**

(iii) The amount of interest accrued in the first year is $(100000)(0.06) = 6000$.



 

## Example 5

*A loan of 100,000 is being repaid by 15 equal annual installments made at the end of each year at 6% interest effective annually.*
*(i) Find the amount of each annual installment.*
*(ii) Find the finance charge of this loan.*
*(iii) Find how much interest is accrued in the first year.*
*(iv) Find how principal is repaid in the first payment.*
*(v) Find the balance in the loan immediately after the first payment.*

**Solution:**

(iv) The amount of principal repaid in the first year is $10296.2764 - 6000 = 4296.2764$.

## Example 5

*A loan of 100,000 is being repaid by 15 equal annual installments made at the end of each year at 6% interest effective annually.*
*(i) Find the amount of each annual installment.*
*(ii) Find the finance charge of this loan.*
*(iii) Find how much interest is accrued in the first year.*
*(iv) Find how principal is repaid in the first payment.*
*(v) Find the balance in the loan immediately after the first payment.*

**Solution:**

(v) The balance in the loan immediately after the first payment is $100000 - 4296.2764 = 95703.7236$.



 

## Example 6

*A loan $L$ is being paid with 20 equal annual payments at the end of each year. The principal portion of the 8–th payment is 827.65 and the interest portion is 873.81. Find $L$.*



 

## Example 6

*A loan L is being paid with 20 equal annual payments at the end of each year. The principal portion of the 8–th payment is 827.65 and the interest portion is 873.81. Find L.*

**Solution:** We know that

$$827.65 = P\nu^{n+1-k} = P\nu^{13}, 873.81 = P(1-\nu^{n+1-k}) = P(1-\nu^{13}).$$

Adding the two equations, we get that $P = 827.65 + 873.81 = 1701.46$. From the equation $827.65 = 1701.46(1+i)^{-13}$, we get that $i = 5.7\%$. Hence, $L = 1701.46 a_{\overline{20}|5.7\%} = 20000$.



 

A way to pay a loan is to pay interest as it accrues and to pay the principal in level installments. Suppose that a loan of amount $L$ is paid at the end of each year for $n$ years. At the end of each year two payments are made: one paying the interest accrued and another one making a principal payment of $\frac{L}{n}$. At the end of $j$ years the outstanding balance is $\frac{L(n-j)}{n}$. Hence, the interest payment at the end of $j$ years is $i\frac{L(n+1-j)}{n}$. The total payment made at the end of the $j$–th year is $\frac{L}{n} + i\frac{L(n+1-j)}{n} = \frac{L}{n}(1 + i(n+1-j))$.



 

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*



 

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(i) Find the amount of each payment of principal.

## Example 7

*Samuel takes a loan of $175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(i) Find the amount of each payment of principal.

**Solution:** (i) The annual payment of principal is $\frac{175000}{15} = 11666.67$.

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(ii) Find the outstanding principal owed at the end of the ninth year.



 

## Example 7

*Samuel takes a loan of $175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(ii) Find the outstanding principal owed at the end of the ninth year.

**Solution:** (ii) The outstanding principal owed at the end of the ninth year is $\frac{(175000)(15-9)}{15} = 70000$.



 

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(iii) Find the interest accrued during the tenth year.

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(iii) Find the interest accrued during the tenth year.

**Solution:** (iii) The amount of interest paid at the end of the tenth year is $(0.085)70000 = 5950$.

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(iv) Find the total amount of payments made at the end of the tenth year.



 

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(iv) Find the total amount of payments made at the end of the tenth year.

**Solution:** (iv) The total amount of payments made at the end of the tenth year is $11666.67 + 5950 = 17616.67$.

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(v) Find the total amount of payments which Samuel makes.



 

## Example 7

*Samuel takes a loan of \$175000. He will pay the loan in 15 years by paying the interest accrued at the end of each year, and paying level payments of the principal at the end of each year. The annual effective rate of interest of the loan is 8.5%.*

(v) Find the total amount of payments which Samuel makes.

**Solution:** (v) The interest payment at the end of $j$ years is $i\frac{L(n+1-j)}{n} = (0.085)\frac{(175000)(16-j)}{15}$. Hence, the total interest payments are

$$(0.085)\sum_{j=1}^{15}\frac{(175000)(16-j)}{15}$$
$$=(0.085)\frac{175000}{15}\left((16)(15)-\frac{(16)(15)}{2}\right)=119000.$$

The total amount of payments which Samuel makes is $119000 + 175000 = 294000$.



 

## Example 8

*A loan of \$150000 is going to be paid over 20 years with monthly payments. The first payment is one month from now. During each year, the payments are constant. But, they increase by 3% each year. The annual effective rate of interest is 6%. Calculate the total amount of the payments made during the first year. Calculate the outstanding loan balance on the loan ten years from now after the payment is made.*



 

**Solution:** Let $P$ be the monthly payment during the first year. During the $k$–th year, 12 payments of $P(1.03)^{k-1}$ are made. The value of these payments at the end of the $k$–th year is $P(1.03)^{k-1}s_{\overline{12}|i^{(12)}/12} = P(1.03)^{k-1}12.32652834$, where we have used that $i^{(12)} = 5.84106068\%$. So, the cashflow of payments is equivalent to

| Payments | $P12.3265$ | $P(1.03)12.3265$ | $\cdots$ | $P(1.03)^{19}12.3265$ |
|---|---|---|---|---|
| Time | 1 | 2 | $\cdots$ | 20 |

The present value of this cashflow is the loan amount:

$$150000 = \frac{12.3265P}{1+r}a_{\overline{n}|\frac{i-r}{1+r}} = \frac{12.3265P}{1.03}a_{\overline{20}|\frac{0.06-0.03}{1+0.03}} = 179.493145P$$

and $P = 835.6865105$.

The outstanding loan balance on the loan ten years from now after the payment is made is

$$(835.6865105)(1.03)^9(12.3265)a_{\overline{10}|\frac{0.06-0.03}{1+0.03}} = 201586.9934.$$



 

## Example 9

*Mary takes on a loan of \$135,000. The loan is being repaid by a 10–year increasing annuity–immediate. The initial payment is* 10000, *and each subsequent payment is* $x$ *larger than the preceding payment. The annual effective interest rate is* 6.5%. *Determine the principal outstanding immediately after the 5–th payment.*



 

**Solution:** The cashflow is

| Contributions | 10000 | 10000 + $x$ | 10000 + $2x$ | $\cdots$ | 10000 + $9x$ |
|---|---|---|---|---|---|
| Time | 1 | 2 | 3 | $\cdots$ | 10 |

The present value of the payments is

$$135000 = (10000 - x)a_{\overline{10}|6.5\%} + x(Ia)_{\overline{10}|6.5\%}$$
$$=(10000 - x)(7.188830223) + 35.82836665x.$$

So, $x = \frac{135000-(7.188830223)(10000)}{35.82836665-7.188830223} = 2203.656401.$
The payments to be made after the 5-th payment are

| Contributions | 10000 + (5)(2203.656401) | $\cdots$ | 10000 + (9)(2203.656 |
|---|---|---|---|
| Time | 6 | $\cdots$ | 10 |

Its present value at time 5 is

$$(10000 + (4)(2203.656401))a_{\overline{5}|6.5\%} + (2203.656401)(Ia)_{\overline{5}|6.5\%}$$
$$=78187.55276 + 26321.63894 = 104509.1917.$$



 