Manual for SOA Exam FM/CAS Exam 2.

Chapter 4. Amortization and sinking bonds.

Section 4.2. Sinking funds.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2,
Financial Mathematics. Spring 2009 Edition",
available at http://www.actexmadriver.com/

An alternate way to repay a loan is to make two payments, one directly to the lender and another to an auxiliary fund. The auxiliary fund is called a **sinking fund**. The payments made directly to the lender apply to the principal. The deposits made into the sinking fund do not. Usually, the sinking fund accumulates with a different interest rate than the rate charged by the lender. At the end of the duration of the loan, the borrower withdraws the total accumulated in the sinking fund and uses this money to pay the loan to the lender.

Usually, we consider the case of payments made at the end of each of n periods. The simplest case is the one when all the payments are equal. Let i be the periodic effective rate charged by the lender on the loan. At the end of each period, the borrower pays P directly to lender. The borrower deposits Q into a sinking fund earning a rate of interest j. Usually, $j < i$. The cashflow of payments to the principal is

Contributions	0	P	P	$\cdots$	P	$P+R$
Time	0	1	2	$\cdots$	$n-1$	n

where R is the lump–sum payment obtained by withdrawing the total accumulated in the sinking fund at the end of n periods. The cashflow of deposits in the sinking fund is

Contributions	0	Q	Q	Q	$\cdots$	Q
Time	0	1	2	3	$\cdots$	n

Hence, the accumulated value in the sinking fund at time n is $R = Qs_{\overline{n}|j}$.

The borrower's total cashflow is

Contributions	L	$-P-Q$	$-P-Q$	$-P-Q$	$\cdots$	$-P-Q$
Time	0	1	2	3	$\cdots$	n

The lender cashflow is

Contributions	$-L$	P	P	P	$\cdots$	$P+R$
Time	0	1	2	3	$\cdots$	n

In order to the loan to be repaid:

$$L = Pa_{\overline{n}|i} + R(1+i)^{-n} = Pa_{\overline{n}|i} + Qs_{\overline{n}|j}(1+i)^{-n}.$$

Example 1

Dave borrows 150,000 from a trust company at an annual effective rate of interest of 9.5%. He agrees to pay the interest annually at the end of the each year, and build up a sinking fund which will repay the loan at the end of 15 years. The sinking fund accumulates at annual effective rate of interest of 4.5%. Calculate:

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(i) The annual interest payments.

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(i) The annual interest payments.

Solution: (i) The annual interest payment is $150000(0.095) = 14250$.

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(ii) The annual sinking fund payment.

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(ii) The annual sinking fund payment.

Solution: (ii) $150000 = Qs_{\overline{15}|4.5\%}$ and $Q = 7217.071217$.

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(iii) Dave's total annual outlay.

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(iii) Dave's total annual outlay.

Solution: (iii) Dave's total annual outlay is $14250+7217.071217 = 21467.07122$.

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(iv) The annual effective rate of interest i' for which the payments made at the end of the year will be equal to Dave's total annual outlay.

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(iv) The annual effective rate of interest i' for which the payments made at the end of the year will be equal to Dave's total annual outlay.

Solution: (iv) $150000 = (21467.07122)a_{\overline{15}|i'}$ and $i' = 11.52440895\%$. Notice that although Dave borrows at a rate 9.5%, by getting only 4.5% in his sinking fund, the actual rate of interest Dave which is paying is higher.

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(v) Find $i + \frac{n-1}{n+1}(i-j)$, where $i = 9.5\%$, $j = 4.5\%$ and $n = 15$, and compare with i'.

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(v) Find $i+\frac{n-1}{n+1}(i-j)$, where $i=9.5\%$, $j=4.5\%$ and $n=15$, and compare with i'.
Solution: (v) $i+\frac{n-1}{n+1}(i-j)=0.095+\frac{(15-1)(0.95-0.045)}{15+1}=0.13875$.
which is sort of close to 0.1152440895.

Example 2

Steve repays a loan of \$18,000 by making interest payment at the end of the year for 15 years and equal deposits at the end of each year into a sinking fund for 15 years. At the end of the 15 years, Steve withdraws the balance from the sinking fund and pays the loan. The sinking fund earns 6% effective annually. Immediately after the fourth payment, the yield on the sinking fund increases to 7% effective annually. At that time, Steve adjusts his sinking fund payment to x so that the sinking fund will accumulate to \$18,000, 15 years after the original loan date. Find x.

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Solution: Let Q be the initial payment Joe makes to the sinking fund. Then, $18000 = Qs_{\overline{15}|6\%}$. Hence, $Q = 773.3297512$. The balance in the sinking fund immediately after the fourth payment is $773.3297512 s_{\overline{4}|6\%} = 3383.020703$. The final accumulation in the sinking fund is 18000. So, $18000 = 3383.020703(1.07)^{11} + x s_{\overline{11}|7\%}$ and $x = 689.2751$, which can be found doing:

11	N	7	I/Y	3383.020703	PV	−18000	FV	CPT	PMT

