Manual for SOA Exam FM/CAS Exam 2.

Chapter 6. Variable interest rates and portfolio insurance.
Section 6.3. Term structure of interest rates.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Term structure of interest rates

The relationship between yield and time to mature is called the **term structure of interest rates**.

As larger as money is tied up in an investment as more likely a default is. Usually, interest rates increase with maturity date.

For US Treasury zero–coupons bonds, different interest rates are given according with the maturity date.

Definition 1

A **yield curve** *is a graph that shows interest rates (vertical axis) versus (maturity date) duration of a investment/loan (horizontal axis).*

Yield curves are studied to predict of changes in economic activity (economic growth, inflation, etc.).

Example 1

A bank offers CD's with the following interest rates

length of the investment (in years)	*1 year*	*2 years*	*3 years*	*4 years*	*5 years*
Interest rate	7%	8%	8.5%	9%	9.25%

Graph the yield curve for these interest rates.

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Graph the yield curve for these interest rates.

Solution: Assuming that the interest rate is a linear function between the given points, the yield curve is given by the graph in Figure 1.

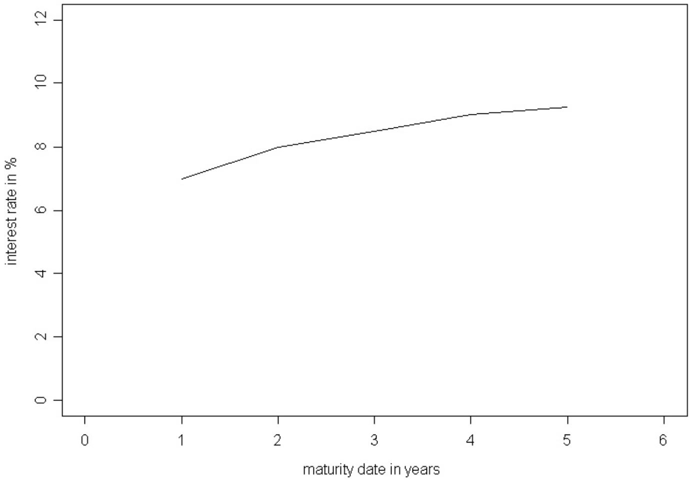

Figure 1: Yield curve for Example 1

The interest rates appearing in the yield curve are called the **spot rates**. Thus, for Example 1, the spot rates are 7%, 8%, 8.5%, 9% and 9.25%.

Definition 2

*The j **year spot rate** s_j is the rate of interest charged in a loan paid with a unique payment at the end of j years.*

Note that is the j **year spot rate** s_j is as an effective annual rate of interest, the current j year interest factor is $(1 + s_j)^j$. Money invested now multiply by $(1 + s_j)^j$ in j years. The price of a zero coupon j–year bond with face value F is $P = F(1 + s_j)^{-j}$.

Spot rates refer to a fixed maturity date. Usually, bonds have coupon payments over time. But, often **strip bonds** are traded. Strip or **zero coupon bonds** are bonds that have being "separated" into their component parts (each coupon payment and the face value). Often strip bonds are obtained from US Treasury bonds. A financial trader (strips) "separates" the coupons from a US Treasury bond, by accumulating a large number of US Treasury bonds and selling the rights of obtaining a particular payment to an investor. In this way, the investor can buy a strip bond as an individual security. The strip bond market consists of coupons and residuals, with coupons representing the interest portion of the original bond and the residual representing the principal portion. An investor will get a unique payment from a strip bond. In this situation, interest rates of a strip bond depend on the maturity date. The yield rate of a zero–coupon bond is called its spot rate.

The following table consists of the Daily Treasury Yield Curve Rates, which can be found at
http://www.treas.gov/offices/domestic-finance/debt-management/interest-rate/yield.html

Date	1 mo	3 mo	6 mo	1 yr	2 yr	3 yr	5 yr	7 yr	10 yr
07/01/04	1.01	1.22	1.64	2.07	2.64	3.08	3.74	4.18	4.57
07/02/04	1.07	1.30	1.61	2.02	2.54	2.96	3.62	4.08	4.48
07/06/04	1.11	1.34	1.68	2.15	2.56	2.99	3.65	4.10	4.49
07/07/04	1.16	1.30	1.64	2.00	2.56	2.99	3.67	4.10	4.50
07/08/04	1.14	1.27	1.63	1.99	2.55	2.97	3.65	4.09	4.49
07/09/04	1.14	1.28	1.63	2.00	2.55	2.96	3.64	4.08	4.49

Suppose that a zero–coupon bond with a face value of F and maturity j years has a price of P_j. Then, $P_j(1+s_j)^j = F$, where s_j is the j year spot rate. Note that a payment of P_j now is exchanged by a payment of F in j years. $(1+s_j)^j$ is the interest factor from year zero to year j. If the current interest rates follow the accumulation function is $a(t)$, $t \geq 0$, then $a(j) = (1+s_j)^j$, i.e. $s_j = (a(j))^{1/j} - 1$.

Example 2

The following table lists prices of zero-coupon \$1000 bonds with their respective maturities:

Number of years to maturity	Price
1	\$980.39
2	\$957.41
5	\$888.18
10	\$781.20

Calculate the 1–year, 2–year, 5–year, and 10–year spot rates of interest.

Example 2

The following table lists prices of zero-coupon $1000 bonds with their respective maturities:

Number of years to maturity	*Price*
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2	*$957.41*
5	*$888.18*
10	*$781.20*

Calculate the 1–year, 2–year, 5–year, and 10–year spot rates of interest.

Solution: Since $(1000)(1+s_1)^{-1} = 980.39$, $(1000)(1+s_2)^{-2} = 957.41$, $(1000)(1+s_5)^{-5} = 888.17$, and $(1000)(1+s_{10})^{-10} = 781.1984$, we get $s_1 = 2.00\%$, $s_2 = 2.20\%$, $s_5 = 2.40\%$, $s_{10} = 2.50\%$.

The present value at time zero of a cashflow

Contributions	0	C_1	C_2	$\cdots$	C_n
Time	0	1	2	$\cdots$	n

following the spot rates

spot rate	s_1	s_2	$\cdots$	s_n
maturity time	1	2	$\cdots$	n

is given by the formula

$$PV = \sum_{j=1}^{n} (1+s_j)^{-j} C_j.$$

Example 3

(i) Find the price of a 2–year 1000 *par value* 6% *bond with semi–annual coupons using the spot rates:*

nominal annual interest rate convertible semiannually	4%	5%	6%	7%
maturity time (in half years)	1	2	3	4

Example 3

(i) Find the price of a 2–year 1000 par value 6% bond with semi–annual coupons using the spot rates:

nominal annual interest rate convertible semiannually	4%	5%	6%	7%
maturity time (in half years)	1	2	3	4

Solution: (i) The cashflow of payments is

Payments	30	30	30	1030
Time (in half years)	1	2	3	4

The present value of these payments is

$$PV = (30)\left(1+\frac{0.04}{2}\right)^{-1} + (30)\left(1+\frac{0.05}{2}\right)^{-2} + (30)\left(1+\frac{0.06}{2}\right)^{-3} + (1030)\left(1+\frac{0.07}{2}\right)^{-4}$$

$$=29.41176 + 28.55443 + 27.45425 + 897.5855 = 983.0059$$

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(ii) Find the annual effective yield rate of the previous bond, if bought at the price in (i).

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nominal annual interest rate convertible semiannually	4%	5%	6%	7%
maturity time (in half years)	1	2	3	4

(ii) Find the annual effective yield rate of the previous bond, if bought at the price in (i).

Solution: (ii) To find the yield rate, we solve for $i^{(2)}$ in $983.0059 = 30a_{\overline{4}|i^{(2)}/2} + 1000(1 + i^{(2)}/2)^{-4}$, to get $i^{(2)} = 6.92450\%$. The annual effective yield rate is $i = 7.0443\%$.

Note that $i^{(2)} = 6.92450\%$ is a sort of average of the spot rates used to find the price of the bond. Since the biggest payment is at time $t = 4$ half years, $i^{(2)} = 6.92450\%$ is close to 7%.

The **one year forward rate for the j–th year** f_j is defined as

$$f_j = \frac{(1+s_j)^j}{(1+s_{j-1})^{j-1}} - 1.$$

f_j is also called the 1 year forward rate from time $j-1$ to time j.
f_j is also called the 1 year forward rate from the j–th year.
f_j is also called the $(j-1)$–year forward rate.
f_j is also called the $(j-1)$–year deferred 1–year forward rate.
f_j is also called the $(j-1)$–year forward rate, 1–year interest rate.
$1+f_j$ is the interest factor from year $j-1$ to year j.

$(1+s_{j-1})^{j-1}$ is the interest factor from year zero to year $j-1$.
$(1+s_j)^j$ is the interest factor from year zero to year j.
Hence, we have that $(1+s_{j-1})^{j-1}(1+f_j)=(1+s_j)^j$.
Notice that by the definition of the one year forward rate

$$
\begin{aligned}
&(1+s_2)^2=(1+s_1)(1+f_2),\\
&(1+s_3)^3=(1+s_2)^2(1+f_3)=(1+s_1)(1+f_2)(1+f_3),\\
&(1+s_4)^4=(1+s_3)^3(1+f_4)=(1+s_1)(1+f_2)(1+f_3)(1+f_4).
\end{aligned}
$$

In general,

$$(1+s_n)^n=(1+s_1)(1+f_2)(1+f_3)\cdots(1+f_n).$$

Example 4

Suppose that the following spot rates are given:

maturity time (in years)	1	2	3	4	5
Interest rate	12.00%	11.75%	11.25%	10.00%	9.25%

Calculate the one–year forward rates for years 2 through 5.

Example 4

Suppose that the following spot rates are given:

maturity time (in years)	1	2	3	4	5
Interest rate	12.00%	11.75%	11.25%	10.00%	9.25%

Calculate the one–year forward rates for years 2 through 5.

Solution:

$$f_2 = \frac{(1.1175)^2}{1.12} - 1 = 0.115006$$
$$f_3 = \frac{(1.1125)^3}{(1.1175)^2} - 1 = 0.102567$$
$$f_4 = \frac{(1.1)^4}{(1.1125)^3} - 1 = 0.063336$$
$$f_5 = \frac{(1.0925)^5}{(1.1)^4} - 1 = 0.063008$$

The one–year forward rate for year 2 is 11.5006%.
The one–year forward rate for year 3 is 10.2567%.
The one–year forward rate for year 4 is 6.3336%.
The one–year forward rate for year 5 is 6.3008%.

Example 5

The following table lists prices of zero-coupon $100 bonds with their respective maturities:

Number of years to maturity	Price
1	$96.15
2	$92.10
3	$87.63
4	$82.27

Example 5

The following table lists prices of zero-coupon \$100 bonds with their respective maturities:

Number of years to maturity	*Price*
1	*\$96.15*
2	*\$92.10*
3	*\$87.63*
4	*\$82.27*

(i) Calculate the 1–year, 2–year, 3–year, and 4–year spot rates of interest.

Example 5

The following table lists prices of zero-coupon $100 bonds with their respective maturities:

Number of years to maturity	*Price*
1	*$96.15*
2	*$92.10*
3	*$87.63*
4	*$82.27*

(i) Calculate the 1–year, 2–year, 3–year, and 4–year spot rates of interest.

Solution: (i) Note that the price of j–th bond is $P_j = (100)(1+s_j)^{-j}$. To get the j year spot rate s_j, we solve $(100)(1+s_1)^{-1} = 96.15$, $(100)(1+s_2)^{-2} = 92.10$, $(100)(1+s_3)^{-3} = 87.63$, and $(100)(1+s_4)^{-4} = 82.27$, to get $s_1 = 4.00\%$, $s_2 = 4.20\%$, $s_3 = 4.50\%$, $s_4 = 5.00\%$.

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The following table lists prices of zero-coupon \$100 bonds with their respective maturities:

Number of years to maturity	*Price*
1	*\$96.15*
2	*\$92.10*
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(ii) Calculate the 1–year, 2–year, and 3–year forward rates of interest.

Example 5

The following table lists prices of zero-coupon $100 bonds with their respective maturities:

Number of years to maturity	*Price*
1	*$96.15*
2	*$92.10*
3	*$87.63*
4	*$82.27*

(ii) Calculate the 1–year, 2–year, and 3–year forward rates of interest.

Solution: (ii) To get the $j-1$ year forward rate f_j, we do $f_j = \frac{(1+s_j)^j}{(1+s_{j-1})^{j-1}} - 1 = \frac{P_{j-1}}{P_j} - 1$, where P_j is the price of the j–th bond. We get that:

$$f_2 = \frac{96.15}{92.10} - 1 = 4.397394\%,$$
$$f_3 = \frac{92.10}{87.63} - 1 = 5.100993\%,$$
$$f_4 = \frac{87.63}{82.27} - 1 = 6.515133\%.$$

