Manual for SOA Exam FM/CAS Exam 2.

Chapter 6. Variable interest rates and portfolio insurance.

Section 6.4. Duration, convexity.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
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Duration

Next we will assume that the rate of interest is constant over maturity.

Definition 1

The **duration** *(or Macaulay's duration) of a cashflow*

Contributions	C_1	C_2	$\cdots$	C_n
Time in years	1	2	$\cdots$	n

with $C_j \geq 0$ *for each* $1 \leq j \leq m$, *is defined as*

$$\bar{d} = \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \frac{\sum_{j=1}^{n} jC_j(1+i)^{-j}}{\sum_{j=1}^{n} C_j(1+i)^{-j}} = \sum_{j=1}^{n} j\frac{C_j\nu^j}{\sum_{k=1}^{n} C_k\nu^k}.$$

Main Properties of volatility

- The duration is an average of the times when the payments of the cashflow are made:

$$\bar{d} = \sum_{j=1}^{n} j \frac{C_j \nu^j}{\sum_{k=1}^{n} C_k \nu^k} = \sum_{j=1}^{n} j w_j,$$

  where $w_j = \frac{C_j \nu^j}{\sum_{k=1}^{n} C_k \nu^k}$ satisfy $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.
- w_j is the fraction of the present value of contribution at time t over the present value of the whole cashflow.
- If $C_{j_0} > 0$ and $C_j = 0$, for each $j \neq j_0$, then $\bar{d} = j_0$, for each rate of interest i.
- The units of the duration are years.
- The Macaulay duration is a measure of the price sensitivity of a cashflow to interest rate changes.

Example 1

An investment pays 1000 at the end of year two and 1000 at the end of year 12. The annual effective rate of interest is 8%. Calculate the Macaulay duration for this investment.

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Solution:

$$\bar{d} = \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \frac{(2)(1000)(1.08)^{-2} + (12)(1000)(1.08)^{-12}}{(1000)(1.08)^{-2} + (1000)(1.08)^{-12}}$$

$$= 5.165633881 \text{ years.}$$

Theorem 1

Let $r > 0$. If the Macaulay duration of the cashflow

Contributions	C_1	C_2	$\cdots$	C_n
Time in years	1	2	$\cdots$	n

is $\bar{d}$, then the Macaulay duration of the cashflow

Contributions	rC_1	rC_2	$\cdots$	rC_n
Time in years	1	2	$\cdots$	n

is $\bar{d}$.

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Time in years	1	2	$\cdots$	n

is $\bar{d}$.

Proof: The duration of the modified cashflow is

$$\frac{\sum_{j=1}^{n} jrC_j\nu^j}{\sum_{j=1}^{n} rC_j\nu^j} = \frac{\sum_{j=1}^{n} jC_j\nu^j}{\sum_{j=1}^{n} C_j\nu^j} = \bar{d}.$$

Example 2

The Macaulay duration of a 10–year annuity–immediate with annual payments of \$1000 is 5.6 years. Calculate the Macaulay duration of a 10–year annuity–immediate with annual payments of \$50000.

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Solution: Both cashflows have duration 5.6 years.

Theorem 2

If the Macaulay duration of the cashflow

Contributions	C_1	C_2	$\cdots$	C_n
Time in years	1	2	$\cdots$	n

is $\bar{d}$*, then the Macaulay duration of the cashflow*

Contributions	C_1	C_2	$\cdots$	C_n
Time in years	$t+1$	$t+2$	$\cdots$	$t+n$

is $\bar{d}+t$.

Theorem 2

If the Macaulay duration of the cashflow

Contributions	C_1	C_2	$\cdots$	C_n
Time in years	1	2	$\cdots$	n

is $\bar{d}$, then the Macaulay duration of the cashflow

Contributions	C_1	C_2	$\cdots$	C_n
Time in years	$t+1$	$t+2$	$\cdots$	$t+n$

is $\bar{d}+t$.

Proof:

$$\frac{\sum_{j=1}^{n}(t+j)C_j\nu^j}{\sum_{j=1}^{n}C_j\nu^j} = \frac{t\sum_{j=1}^{n}C_j\nu^j + \sum_{j=1}^{n}jC_j\nu^j}{\sum_{j=1}^{n}C_j\nu^j} = t + \frac{\sum_{j=1}^{n}jC_j\nu^j}{\sum_{j=1}^{n}C_j\nu^j}.$$

Example 3

The Macaulay duration of a 10–year annuity–immediate with annual payments of \$1000 is 5.6 years. Calculate the Macaulay duration of a 10–year annuity–due with annual payments of \$5000.

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The Macaulay duration of a 10–year annuity–immediate with annual payments of \$1000 is 5.6 years. Calculate the Macaulay duration of a 10–year annuity–due with annual payments of \$5000.

Solution: The Macaulay duration of the two annuities does not dependent on the amount of the payment. So, we may assume that the two annual payments agree. Since the cashflow of an annuity–due is obtained from the cashflow of an annuity–immediate by translating payments 1 year, the answer is $5.6 - 1 = 4.6$ years.

Theorem 3

Suppose that two cashflows have durations $\bar{d}_1$ and $\bar{d}_2$, respectively, present values P_1 and P_2, respectively. Then, the duration of the combined cashflow is

$$\bar{d} = \frac{P_1\bar{d}_1 + P_2\bar{d}_2}{P_1 + P_2}.$$

By induction the previous formula holds for a combination of finitely many cashflows. Suppose that we have n cashflows. The j–the cashflow has present value P_j and duration $\bar{d}_j$. Then, the duration of the combined cashflow is

$$\bar{d} = \frac{\sum_{j=1}^{n} P_j(i)\bar{d}_j}{\sum_{j=1}^{n} P_j(i)}.$$

Proof: Suppose that the considered cashflows are

Contributions	0	C_1	C_2	$\cdots$	C_n
Time	0	1	2	$\cdots$	n

and

Contributions	0	D_1	D_2	$\cdots$	D_n
Time	0	1	2	$\cdots$	n

Then, the combined cashflow is

Contributions	0	$C_1 + D_1$	$C_2 + D_2$	$\cdots$	$C_n + D_n$
Time	0	1	2	$\cdots$	n

We have that $P_1 = \sum_{j=1}^n C_j \nu^j$ and $P_2 = \sum_{j=1}^n D_j \nu^j$. By definition of duration,

$$\bar{d}_1 = \frac{\sum_{j=1}^n jC_j \nu^j}{\sum_{j=1}^n C_j \nu^j} = \frac{\sum_{j=1}^n jC_j \nu^j}{P_1}$$

and

$$\bar{d}_2 = \frac{\sum_{j=1}^n jD_j \nu^j}{\sum_{j=1}^n D_j \nu^j} = \frac{\sum_{j=1}^n jD_j \nu^j}{P_2}.$$

Hence,

$$\bar{d} = \frac{\sum_{j=1}^n j(C_j + D_j)\nu^j}{\sum_{j=1}^n (C_j + D_j)\nu^j} = \frac{\sum_{j=1}^n jC_j \nu^j + \sum_{j=1}^n jD_j \nu^j}{\sum_{j=1}^n C_j \nu^j + \sum_{j=1}^n D_j \nu^j} = \frac{\bar{d}_1 P_1 + \bar{d}_2 P_2}{P_1 + P_2}.$$

Example 4

An insurance has the following portfolio of investments:
(i) Bonds with a value of \$1,520,000 and duration 4.5 years.
(ii) Stock dividends payments with a value of \$1,600,000 and duration 14.5 years.
(iii) Certificate of deposits payments with a value of \$2,350,000 and duration 2 years.
Calculate the duration of the portfolio of investments.

Example 4

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(ii) Stock dividends payments with a value of \$1,600,000 and duration 14.5 years.
(iii) Certificate of deposits payments with a value of \$2,350,000 and duration 2 years.
Calculate the duration of the portfolio of investments.

Solution: The duration of the portfolio is

$$\bar{d} = \frac{\sum_{j=1}^{n} P_j(i)\bar{d}_j}{\sum_{j=1}^{n} P_j(i)}$$
$$= \frac{(4.5)(1520000) + (14.5)(1600000) + (2)(2350000)}{1520000 + 1600000 + 2350000}$$
$$= 6.351005484 \text{ years.}$$

Theorem 4

The Macaulay duration of a level payments annuity–immediate is

$$\bar{d} = \frac{(Ia)_{\overline{n|}i}}{a_{\overline{n|}i}}.$$

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$$\bar{d} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}}.$$

Proof.

We have that

$$\bar{d} = \frac{\sum_{j=1}^{n} jP\nu^j}{\sum_{j=1}^{n} P\nu^j} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}}.$$

$\square$

Example 5

Calculate Macaulay the duration of a 15–year annuity immediate with level payments if the current effective interest rate per annum is 5%.

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Solution: The Macaulay the duration is

$$\bar{d} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}} = \frac{(Ia)_{\overline{15}|5\%}}{a_{\overline{15}|5\%}} = \frac{73.66768937}{10.37965804} = 7.097313716.$$

Theorem 5

The duration of a level payments perpetuity–immediate is

$$\bar{d} = \frac{1+i}{i}.$$

Proof.

We have that

$$\bar{d} = \frac{\sum_{j=1}^{\infty} jP\nu^j}{\sum_{j=1}^{\infty} P\nu^j} = \frac{(Ia)_{\overline{\infty}|i}}{a_{\overline{\infty}|i}} = \frac{\frac{1+i}{i^2}}{\frac{1}{i}} = \frac{1+i}{i}.$$

$\square$

Example 6

Suppose that the Macaulay duration of a perpetuity immediate with level payments of 1000 at the end of each year is 21. *Find the current effective rate of interest.*

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Suppose that the Macaulay duration of a perpetuity immediate with level payments of 1000 at the end of each year is 21*. Find the current effective rate of interest.*

Solution: We have that $21 = \bar{d} = \frac{1+i}{i}$. So, $i = \frac{1}{20} = 5\%$.

Theorem 6

The duration of n year bond with r% annual coupons, face value F and redemption value C is

$$\bar{d} = \frac{Fr(Ia)_{\overline{n}|i} + Cn\nu^n}{Fra_{\overline{n}|i} + C\nu^n}.$$

Proof.

Since the cashflow

Contributions	Fr	Fr	$\cdots$	Fr	$Fr + C$
Time	1	2	$\cdots$	$n-1$	n

the duration is

$$\bar{d} = \frac{Fr\sum_{j=1}^{n} j\nu^j + Cn\nu^n}{Fr\sum_{j=1}^{n} \nu^j + C\nu^n} = \frac{Fr(Ia)_{\overline{n}|i} + Cn\nu^n}{Fra_{\overline{n}|i} + C\nu^n}.$$

Example 7

Megan buys a 10–year 1000–face–value bond with a redemption value of 1200 which pay annual coupons at rate 7.5%. Calculate the Macaulay duration if the effective rate of interest per annum is 8%.

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Solution: We have that

$$\bar{d} = \frac{Fr(Ia)_{\overline{n}|i} + Cn\nu^n}{Fra_{\overline{n}|i} + C\nu^n}$$

$$= \frac{(1000)(0.075)(Ia)_{\overline{10}|8\%} + (1200)(10)(1.08)^{-10}}{(1000)(0.075)a_{\overline{10}|8\%} + (1200)(1.08)^{-10}}$$

$$= \frac{(75)(32.68691288) + 5558.321857}{(75)(6.710081399) + 555.8321857} = 7.562958059.$$

We have the following table:

Cashflow	Duration $\bar{d}$
zero–coupon bond	$\bar{d} = n$
level payments annuity–immediate	$\bar{d} = \frac{(Ia)_{\overline{n}\mid i}}{a_{\overline{n}\mid i}}$
level payments perpetuity–immediate	$\bar{d} = \frac{1+i}{i}$
regular bond	$\bar{d} = \frac{Fr(Ia)_{\overline{n}\mid i}+Cn\nu^n}{Fra_{\overline{n}\mid i}+C\nu^n}$

Volatility

Consider a cashflow

Contributions	C_1	C_2	$\cdots$	C_n
Time in years	1	2	$\cdots$	n

Definition 2

The quantity $\bar{\nu} = -\frac{d \ln P(i)}{di} = -\frac{P'(i)}{P(i)}$ *is called the* **volatility** *or* **modified duration**.

Notice that $P(i) = \sum_{j=1}^{n} C_j \nu^j = \sum_{j=1}^{n} C_j (1+i)^{-j}$ and $P'(i) = \sum_{j=1}^{n} C_j(-j)(1+i)^{-j-1}$. So,

$$\bar{\nu} = \frac{\sum_{j=1}^{n} j C_j \nu^{j+1}}{\sum_{j=1}^{n} C_j \nu^j}.$$

Since

$$\bar{d} = \frac{\sum_{j=1}^{n} jC_j \nu^j}{\sum_{j=1}^{n} C_j \nu^j}$$

and

$$\bar{\nu} = \frac{\sum_{j=1}^{n} jC_j \nu^{j+1}}{\sum_{j=1}^{n} C_j \nu^j},$$

$$\bar{\nu} = \nu \bar{d}.$$

Main Properties of volatility

- $\bar{\nu} = -\frac{P'(i)}{P(i)}$.
- The volatility measures the loss of present value of the cashflow as i increases relative to the PV of the cashflow.
- $$\bar{\nu} = \frac{\sum_{j=1}^{n} jC_j \nu^{j+1}}{\sum_{j=1}^{n} C_j \nu^j}.$$
- If $C_j \geq 0$, for each $1 \leq j \leq n$, $\bar{\nu} > 0$.
- $\bar{\nu} = \nu \bar{d}$.
- Volatility is measured in years.

The present value $P(i)$ of the above cashflow as a function on i, i.e.

$$P(i) = \sum_{j=1}^{n} C_j \nu^j = \sum_{j=1}^{n} C_j (1+i)^{-j}.$$

It is easy to see that

$$P'(i) = \sum_{j=1}^{n} C_j (-j)(1+i)^{-j-1}, \text{ and } P''(i) = \sum_{j=1}^{n} C_j j(j+1)(1+i)^{-j-2}.$$

If $C_j > 0$, for each $1 \leq j \leq n$, then $P'(i) < 0$ and $P''(i) > 0$, for each $i \geq 0$. This implies that $P(i)$ is a decreasing convex function on i.

Since $P(i)$, $i \geq 0$, is a decreasing function of i, so is $\ln P(i)$. Hence,

$$0 < -\frac{d \ln P(i)}{di} = -\frac{P'(i)}{P(i)} = -\frac{\sum_{j=1}^{n} C_j(-j)(1+i)^{-j-1}}{\sum_{j=1}^{n} C_j(1+i)^{-j}}$$
$$= \frac{\sum_{j=1}^{n} jC_j\nu^{j+1}}{\sum_{j=1}^{n} C_j\nu^{j}}.$$

Hence, $\bar{\nu} > 0$.

Since $\bar{\nu} = \nu \bar{d}$, we have that the volatility satisfies some of the properties of the duration. Suppose that we have n cashflows. The j–the cashflow has present value P_j and duration $\bar{\nu}_j$. Then, the duration of the combined cashflow is

$$\bar{\nu} = \frac{\sum_{j=1}^{n} P_j(i)\bar{\nu}_j}{\sum_{j=1}^{n} P_j(i)}.$$

Example 8

A perpetuity pays 100 immediately. Each subsequent payment in increased by inflation. The current annual effective rate of interest is 6.5%. Calculate the modified duration of the perpetuity assuming that inflation will be 5% annually.

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Solution: The present value of the perpetuity is $P(i) = \frac{100}{i-0.05}$, if $i > 0.05$. Hence, $P'(i) = \frac{-100}{(i-0.05)^2}$,
$\bar{\nu} = -\frac{P'(0.065)}{P(0.065)} = \frac{1}{0.065-0.05} = 66.66666667$.

Example 9

A portfolio consists of four bonds. The prices and modified durations of the four bonds are given by the table:

Bond	Present value	Modified duration in years
Bond A	$15050	4.3
Bond B	$10350	10.4
Bond C	$67080	7.6
Bond D	$16750	6.5

Find the volatility of the whole portfolio.

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Bond D	\$16750	6.5

Find the volatility of the whole portfolio.

Solution: We have that

$$\bar{\nu} = \frac{\sum_{j=1}^{n} P_j(i)\bar{\nu}_j}{\sum_{j=1}^{n} P_j(i)}$$

$$= \frac{(15050)(4.3) + (10350)(10.4) + (67080)(7.6) + (16750)(6.5)}{15050 + 10350 + 67080 + 16750}$$

$$= 7.241948183 \quad \text{years.}$$

Let $P(i)$ be the present value of a portfolio, when i is the effective rate of interest. By a Taylor expansion, for h close to zero,

$$P(i+h) \approx P(i) + P'(i)h = P(i)\left(1 - \nu \bar{d} h\right) = P(i)\left(1 - \bar{\nu} h\right).$$

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Example 10

A portfolio of bonds is worth 535000 at the current rate of interest of 4.75%. Its Macaulay duration is 6.375. Estimate the value of the portfolio if interest rates decrease by 0.10%.

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Example 10

A portfolio of bonds is worth 535000 at the current rate of interest of 4.75%. Its Macaulay duration is 6.375. Estimate the value of the portfolio if interest rates decrease by 0.10%.

Solution: We have that $P(i+h) \approx P(i)\left(1 - \nu\bar{d}h\right)$. In our case,

$$P(0.0475 - 0.0010) \approx 535000(1 - (1.0475)^{-1}(6.375)(-0.001))$$
$$= 538255.9666.$$

If interest rates change from i into $i + h$, the percentage of change in the present value of the portfolio is

$$\frac{P(i+h) - P(i)}{P(i)} \approx \frac{P(i) + P'(i)h - P(i)}{P(i)} = -\nu\bar{d}h = -\bar{\nu}h.$$

If interest rates change from i into $i+h$, the percentage of change in the present value of the portfolio is

$$\frac{P(i+h)-P(i)}{P(i)} \approx \frac{P(i)+P'(i)h-P(i)}{P(i)} = -v\bar{d}h = -\bar{\nu}h.$$

Example 11

A bond has a volatility of 4.5 years, at the current annual interest rate of 5%. Calculate the percentage of loss of value of the bond if the annual effective interest rate increase 250 basis points.

If interest rates change from i into $i+h$, the percentage of change in the present value of the portfolio is

$$\frac{P(i+h)-P(i)}{P(i)} \approx \frac{P(i)+P'(i)h-P(i)}{P(i)} = -\nu\bar{d}h = -\bar{\nu}h.$$

Example 11

A bond has a volatility of 4.5 years, at the current annual interest rate of 5%. Calculate the percentage of loss of value of the bond if the annual effective interest rate increase 250 basis points.

Solution: The percentage of change is $-\bar{\nu}h = -(4.5)(0.025) = -0.1125 = -11.25\%$. The bond loses 11.25% of its value.

Duration is a measurement of how long in years it takes for the payments to be made. Mainly, we will consider applications to the bond market. Duration is an important measure for investors to consider, as bonds with higher durations are riskier and have a higher price volatility than bonds with lower durations. We have the following rules of thumb:

- Higher coupon rates lead to lower duration.
- Longer terms to maturity usually lead to longer duration.
- Higher yields lead to lower duration.

The price of a bond decreases as the rate of interest increases. Suppose that you believe that interest rates will drop soon. You want to make a benefit by buying a bond today and selling it later for a higher price. The profit you make is $P(i+h) - P(i)$, where i is the interest you buy the bond and $i+h$ is the interest rate when you sell the bond. Notice that you make a benefit if $h < 0$. The rate of return in your investment is

$$\frac{P(i+h) - P(i)}{P(i)} \approx -\bar{\nu} h.$$

So, between all possible bonds, you will make a biggest profit investing in the bond with the highest possible volatility.

Example 12

Suppose that you are comparing two five–year bonds with a face value of 1000, and are expecting a drop in yields of 1% almost immediately. The current yield is 8%. Bond 1 has 6% annual coupons and bond 2 has annual 12% coupons. You would like to invest 100,000 in the bond giving you the biggest return.
(i) Which would provide you with the highest potential gain if your outlook for rates actually occurs?
(ii) Find the duration of each bond.

Solution: (i) The price of the bond 1 is

$$(60)a_{\overline{5}|8\%} + 1000(1.08)^{-5} = 920.15.$$

Its price after the change of interest rates is

$$(60)a_{\overline{5}|7\%} + 1000(1.07)^{-5} = 959.00.$$

The gain is 38.85. The percentage of change is $\frac{959.00-920.15}{920.15} = 4.22\%$

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The gain is 38.85. The percentage of change is $\frac{959.00-920.15}{920.15} = 4.22\%$

The price of the bond 2 is

$$(120)a_{\overline{5}|8\%} + 1000(1.08)^{-5} = 1159.71.$$

Its price after the change of interest rates is

$$(120)a_{\overline{5}|7\%} + 1000(1.07)^{-5} = 1205.01.$$

The gain is 45.30. The percentage of change is $\frac{1205.01-1159.71}{1159.71} = 3.91\%$.

Bond 1 is a better investment than bond 2 (if we believe that the interest rates are going to fall to 1%).

(ii) For the first bond, $F = C = 1000$, $Fr = 60$, $i = 8\%$ and $n = 5$. Its price is

$$Fra_{\overline{n}|i} + F\nu^n = 60a_{\overline{5}|8\%} + 1000(1.08)^{-5} = 920.15.$$

and its Macaulay duration is

$$\bar{d} = \frac{Fr(Ia)_{\overline{n}|i} + Fn\nu^n}{Fra_{\overline{n}|i} + F\nu^n} = \frac{60(Ia)_{\overline{5}|0.08} + (1000)(5)(1.08)^{-5}}{920.15}$$

$$= \frac{60(11.3651) + 3402.92}{920.15} = 4.4393.$$

(ii) For the first bond, $F = C = 1000$, $Fr = 60$, $i = 8\%$ and $n = 5$. Its price is

$$Fra_{\overline{n}|i} + F\nu^n = 60a_{\overline{5}|8\%} + 1000(1.08)^{-5} = 920.15.$$

and its Macaulay duration is

$$\bar{d} = \frac{Fr(Ia)_{\overline{n}|i} + Fn\nu^n}{Fra_{\overline{n}|i} + F\nu^n} = \frac{60(Ia)_{\overline{5}|0.08} + (1000)(5)(1.08)^{-5}}{920.15}$$

$$= \frac{60(11.3651) + 3402.92}{920.15} = 4.4393.$$

For the second bond, $F = 1000$, $Fr = 120$, $i = 8\%$ and $n = 5$. Its price is

$$Fra_{\overline{n}|i} + F\nu^n = 120a_{\overline{5}|8\%} + 1000(1.08)^{-5} = 1159.71.$$

and its Macaulay duration is

$$\bar{d} = \frac{120(Ia)_{\overline{5}|0.08} + (1000)(5)(1.08)^{-5}}{1159.71} = \frac{120(11.3651) + 3402.92}{1159.71}$$

$$= 4.1103.$$

Convexity

Definition 3

The **convexity** *of the cashflow*

Contributions	C_1	C_2	$\cdots$	C_n
Time in years	1	2	$\cdots$	n

is defined as

$$\bar{c} = \frac{P''(i)}{P(i)} = \frac{\sum_{j=1}^{n} C_j j(j+1)(1+i)^{-j-2}}{\sum_{j=1}^{n} C_j (1+i)^{-j}} = \frac{\sum_{j=1}^{n} C_j j(j+1)\nu^{-j-2}}{\sum_{j=1}^{n} C_j \nu^{-j}}.$$

Convexity is measured in years^2.

Main Properties of volatility

- Convexity measures the rate of change of the volatility:

$$\frac{d}{di}\bar{\nu} = \frac{d}{di}\frac{P'(i)}{P(i)} = \frac{P''(i)P(i) - P'(i)P'(i)}{(P(i))^2} = \bar{c} - (\bar{\nu})^2.$$

- The second order Taylor expansion of the present value with respect to the yield is:

$$P(i+h) \approx P(i) + P'(i)h + \frac{h^2}{2}P''(i) = P(i)\left(1 - \bar{\nu}h + \frac{h^2}{2}\bar{c}\right).$$

- Convexity is a measure of the curvature of the price–yield curve for a bond. Convexity is related with the second term in the Taylor expansion of the PV.
- Using duration and convexity, we measure of how sensitive the present value of a cashflow is to interest rate changes.

Main Properties of volatility

- Using duration and convexity, we have the following Taylor expansion:

$$P(i+h) \approx P(i)\left(1 - \bar{\nu}h + \frac{h^2}{2}\bar{c}\right).$$

- The percentage change in the PV of a cashflow is

$$\frac{P(i+h) - P(i)}{P(i)} \approx -\bar{\nu}h + \frac{h^2}{2}\bar{c}.$$

- Convexity can be used to compare bonds. If two bonds offer the same duration and yield but one exhibits greater convexity, the bond with greater convexity is more affected by interest rates.

Example 13

A portfolio of bonds is worth 350000 at the current rate of interest of 5.2%. Its modified duration is 7.22. Its convexity is 370. Estimate the value of the portfolio if interest rates increase by 0.2%.

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Solution: We have that $P(i+h) \approx P(i)\left(1 - \bar{\nu} h + \frac{h^2}{2}\bar{c}\right)$. In our case,

$$P(0.052 + 0.002) = 350000\left(1 - (7.22)(0.002) + (370)\frac{(0.002)^2}{2}\right)$$

$=345205.$

Example 14

Calculate the duration, the modified duration and the convexity of a \$5000 face value 15–year zero–coupon bond if the current effective annual rate of interest is 7.5%.

Example 14

Calculate the duration, the modified duration and the convexity of a \$5000 face value 15–year zero–coupon bond if the current effective annual rate of interest is 7.5%.

Solution: Since $P(i) = (5000)(1+i)^{-15}$,
$P'(i) = (5000)(-15)(1+i)^{-16}$,
$P''(i) = (5000)(-15)(-16)(1+i)^{-17}$, we have that
$\bar{\nu} = \frac{-P'(0.75)}{P(0.75)} = (15)(1+0.075)^{-1} = 13.95348837$ years,
$\bar{d} = (1+i)\bar{\nu} = (1.075)(13.95348837) = 15$ years and
$\bar{c} = (-15)(-16)(1+0.75)^{-2} = 78.36734694$ years^2.

Example 15

Calculate the duration, the modified duration and the convexity of a level payments perpetuity–immediate with payments at the end of the year if the current effective annual rate of interest is 5%.

Example 15

Calculate the duration, the modified duration and the convexity of a level payments perpetuity–immediate with payments at the end of the year if the current effective annual rate of interest is 5%.

Solution: Since $P(i)=\frac{C}{i}$, $P'(i)=\frac{-C}{i^2}$, $P''(i)=\frac{2C}{i^3}$, we have that $\bar{\nu}=\frac{-P'(0.05)}{P(0.05)}=\frac{1}{0.05}=20$ years, $\bar{d}=(1+i)\bar{\nu}=(1.05)(20)=21$ ears and $\bar{c}=\frac{2}{i^2}=\frac{2}{(0.05)^2}=800$ years^2.

Example 16

A 100 par value 3 year bond pays annual coupons at a rate 7% coupon rate (with annual coupon payments). The current annual effective interest rate is 7%.

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(i) Calculate the duration, the modified duration and the convexity of the bond.

Solution: (i) The cashflow is

Contributions	7	7	107
Time	1	2	3

The duration is

$$\bar{d} = \frac{(7)(1.07)^{-1} + 2(7)(1.07)^{-2} + 3(107)(1.07)^{-3}}{100} = 2.808018.$$

The modified duration is $\bar{\nu} = \frac{2.808018}{1.07} = 2.6243$. The convexity is

$$\bar{c} = \frac{(7)(1)(2)(1.07)^{-3} + (7)(2)(3)(1.07)^{-4} + (107)(3)(4)(1.07)^{-5}}{100}$$

$=9.58944.$

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A 100 par value 3 year bond pays annual coupons at a rate 7% coupon rate (with annual coupon payments). The current annual effective interest rate is 7%.
(ii) If the interest rate change from 7% to 8%, what is the percentage change in the price of the bond?

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(ii) If the interest rate change from 7% to 8%, what is the percentage change in the price of the bond?

Solution: (ii) If $i = 7\%$, the price of the bond is

$$7a_{\overline{3}|7\%} + 100(1.07)^3 = 100.$$

If $i = 8\%$, the price of the bond is

$$7a_{\overline{3}|8\%} + 100(1.08)^3 = 97.4229.$$

The change in percentage is $\frac{97.4229}{100} - 1 = -2.5771\%$.

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(iii) Using the duration rule, including convexity, what is the percentage change in the bond price?

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Solution: (iii) The estimation in the change in percentage is

$$-\bar{\nu}h + \frac{h^2}{2}\bar{c} = -(2.6243)(0.01) + \frac{(0.01)^2}{2}(9.58944)$$
$$= -0.025760528 = -2.576353\%.$$

Theorem 7

Suppose that we have n different investments. The j–th investment has present value P_j and convexity $\bar{c}_j$. Then, the convexity of the combined investments is

$$\bar{c} = \frac{\sum_{j=1}^{n} P_j(i)\bar{c}_j}{\sum_{j=1}^{n} P_j(i)}.$$

Theorem 7

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Proof.

Notice that

$$\bar{c} = \frac{\sum_{j=1}^{n} P_j''(i)}{P_j(i)} = \frac{\sum_{j=1}^{n} P_j(i)\bar{c}_j}{P_j(i)}.$$

$\square$

Example 17

A company has issued debt using the following bonds:

Bond	Present value	Macaulay's duration	convexity
Bond A	100000	5.3	1.2
Bond B	50000	3.4	3.2
Bond C	120000	12.2	6.2
Bond D	80000	2.3	3.6

Find the Macaulay's duration and the convexity for the entire portfolio.

Solution: Let P_j, $\bar{d}_j$ and $\bar{c}_j$ be the present value, Macaulay's duration and convexity, respectively, of the j–th bond, $1 \leq j \leq 4$. Then, the Macaulay's duration of the whole portfolio is

$$\begin{aligned}\bar{d} &= \frac{\sum_{j=1}^{n} P_j \bar{d}_j}{\sum_{j=1}^{n} P_j}\\ &= \frac{100000(5.3) + 50000(3.4) + 120000(12.2) + 80000(2.3)}{100000 + 50000 + 120000 + 80000}\\ &= 6.708571429.\end{aligned}$$

The convexity of the whole portfolio is

$$\begin{aligned}\bar{c} &= \frac{\sum_{j=1}^{n} P_j \bar{c}_j}{\sum_{j=1}^{n} P_j}\\ &= \frac{100000(1.2) + 50000(3.2) + 120000(6.2) + 80000(3.6)}{100000 + 50000 + 120000 + 80000}\\ &= 3.748571429.\end{aligned}$$

In the case of payments made every $\frac{1}{m}$ years, it is usual to use the nominal rate of interest $i^{(m)}$ as the variable. The present value of the cashflow

Contributions	C_1	C_2	$\cdots$	C_n
Time (in years)	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{n}{m}$

is

$$P(i^{(m)}) = \sum_{j=1}^{n} C_j \left(1 + \frac{i^{(m)}}{m}\right)^{-j}.$$

The duration (or Macaulay's duration) of the cashflow is

$$\bar{d} = \frac{\sum_{j=1}^{n} \frac{j}{m} C_j \left(1 + \frac{i^{(m)}}{m}\right)^{-j}}{\sum_{j=1}^{n} C_j \left(1 + \frac{i^{(m)}}{m}\right)^{-j}} = \frac{1}{m} \frac{\sum_{j=1}^{n} j C_j \left(1 + \frac{i^{(m)}}{m}\right)^{-j}}{\sum_{j=1}^{n} C_j \left(1 + \frac{i^{(m)}}{m}\right)^{-j}} \text{ years.}$$

The volatility is

$$\bar{\nu} = -\frac{P'(i^{(m)})}{P(i^{(m)})} = -\frac{\sum_{j=1}^{n} C_j(-j)(1 + \frac{i^{(m)}}{m})^{-j-1} \frac{1}{m}}{\sum_{j=1}^{n} C_j(1 + \frac{i^{(m)}}{m})^{-j}} = \left(1 + \frac{i^{(m)}}{m}\right)^{-1} \bar{d}.$$

The convexity is

$$\bar{c} = \frac{P''(i^{(m)})}{P(i^{(m)})} = \frac{\sum_{j=1}^{n} C_j(-j)(-j-1)(1 + \frac{i^{(m)}}{m})^{-j-2} \frac{1}{m^2}}{\sum_{j=1}^{n} C_j(1 + \frac{i^{(m)}}{m})^{-j}}$$

$$= \frac{1}{m^2} \frac{\sum_{j=1}^{n} C_j j(j+1)(1 + \frac{i^{(m)}}{m})^{-j-2}}{\sum_{j=1}^{n} C_j(1 + \frac{i^{(m)}}{m})^{-j}}.$$

Example 18

You are given the following information about a bond:

- *The term–to–maturity is 2 years.*
- *The bond has a 9% annual coupon rate, paid semiannually.*
- *The annual bond–equivalent yield–to–maturity is 8%.*
- *The par value is $100.*

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(i) Calculate the current price of the bond.

Solution: (i) Since $F = 100$, $Fr = 4.5$, $i = 4\%$ and $n = 4$, the price is is

$$Fra_{\overline{n}|i} + P\nu^n = (4.5)a_{\overline{4}|4\%} + 100(1.04)^{-4} = 101.8149.$$

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(ii) Calculate the Macaulay duration of the bond.

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(ii) Calculate the Macaulay duration of the bond.

Solution: (ii) The Macaulay duration is (in years)

$$\bar{d} = \frac{1}{2}\frac{Fr(Ia)_{\overline{n}|i} + nF\nu^n}{Fra_{\overline{n}|i} + F\nu^n} = \frac{1}{2}\frac{(4.5)(8.896856) + (4)(100)(0.854804)}{101.8149} = 1.875744.$$

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(iii) Calculate the convexity of the bond.

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(iii) Calculate the convexity of the bond.
Solution: (iii) The convexity is (in years^2)

$$\bar{c} = \frac{1}{4}\times \frac{(4.5)(1)(2)(1.04)^{-3}+(4.5)(2)(3)(1.04)^{-4}+(4.5)(3)(4)(1.04)^{-5}+(104.5)(4)(5)(1.04)^{-6}}{101.8149}$$

$$= \frac{1}{4}\frac{8.0009 + 23.0797 + 44.3841 + 1651.7574}{101.8149} = 4.241083$$

where we have used $\frac{1}{4}$ because the time is in half years.

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(iv) For a 200 basis point increase in yield, determine the amount of error in using duration and convexity to estimate the price change.

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Solution: (iv) We need to find

$$P(i+h)-P(i)-P'(i)h-P''(i)\frac{h^2}{2}=P(i+h)-P(i)\left(1-\bar{\nu}h+\bar{c}\frac{h^2}{2}\right).$$

The price of the bond after the change in interest rates is

$$P(i+h)=Fra_{\overline{n}|i+h}+P(1+i+h)^{-n}=(4.5)a_{\overline{4}|6\%}+100(1.06)^{-4}=94.8023.$$

So, the error is

$$P(i+h)-P(i)\left(1-\bar{\nu}h+\bar{c}\frac{h^2}{2}\right)$$
$$=94.8023-101.8149\left(1-(1.875744)(1.04)^{-1}(0.02)+4.241083\frac{(0.02)^2}{2}\right)$$
$$=94.8023-101.8149(0.9647762)=-3.426292.$$

Example 18

Find the price and Macaulay duration of the following fixed–income securities, given the annual effective rate of interest 4.75% and par value of each bond is \$1,000.

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(i) 3–year bond with 5.00% annual coupons

Solution: (i) We have $F = 1000$, $r = 0.05$, $i = 4.75\%$, $Fr = 50$ and $n = 3$. The price of the bond is

$$Fra_{\overline{n|}i} + F\nu^n = 50a_{\overline{3|}4.75\%} + 1000(1.0475)^{-3} = 1006.84.$$

The Macaulay duration is (in years)

$$\bar{d} = \frac{Fr(Ia)_{\overline{n|}i} + Fn\nu^n}{Fra_{\overline{n|}i} + F\nu^n} = \frac{50(Ia)_{\overline{3|}0.0475} + (1000)(3)(1.0475)^{-3}}{1006.84}$$

$$= \frac{50(5.3875) + 2610.11}{1006.84} = 2.8599.$$

Example 18

Find the price and Macaulay duration of the following fixed–income securities, given the annual effective rate of interest 4.75% and par value of each bond is \$1,000.

(ii) 3–year bond with 5.00% semiannual coupons

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Find the price and Macaulay duration of the following fixed–income securities, given the annual effective rate of interest 4.75% and par value of each bond is $1,000.

(ii) 3–year bond with 5.00% semiannual coupons

Solution: (ii) We have $i = 0.0475$, $i^{(2)} = 0.046949$, $\frac{i^{(2)}}{2} = 0.0234745$, $F = 1000$, $r = 0.025$, $Fr = 25$ and $n = 6$. The price of the bond is

$$25a_{\overline{6}|0.0234745} + 1000(1.0234745)^{-6} = 1008.45.$$

The Macaulay duration is (in years)

$$\bar{d} = \frac{1}{2}\frac{Fr(Ia)_{\overline{n}|i} + Fn\nu^n}{Fra_{\overline{n}|i} + F\nu^n} = \frac{1}{2}\frac{25(Ia)_{\overline{6}|0.0234745} + (1000)(6)(1.0234745)^{-6}}{1008.45}$$

$$= \frac{(12.5)(19.0026) + 2610.11}{1008.45} = 2.823782.$$

Example 18

Find the price and Macaulay duration of the following fixed–income securities, given the annual effective rate of interest 4.75% and par value of each bond is $1,000.

(iii) 3–year bond with 5.00% quarter coupons.

Example 18

Find the price and Macaulay duration of the following fixed–income securities, given the annual effective rate of interest 4.75% and par value of each bond is \$1,000.

(iii) 3–year bond with 5.00% quarter coupons.

Solution: (iii) We have $i = 0.0475$, $i^{(4)} = 0.046677$, $\frac{i^{(4)}}{4} = 0.0116692$, $F = 1000$, $Fr = 12.5$ and $n = 6$. The price of the bond is

$$12.5a_{\overline{12}|0.0116692} + 1000(1.0116692)^{-12} = 1009.25.$$

The Macaulay duration is (in years)

$$\bar{d} = \frac{1}{4}\frac{Fr(Ia)_{\overline{n}|i} + Fn\nu^n}{Fra_{\overline{n}|i} + F\nu^n} = \frac{1}{4}\frac{12.5(Ia)_{\overline{12}|0.0116692} + (1000)(12)(1.0116692)^{-12}}{1009.25}$$

$$= \frac{(3.125)(70.8531) + 2610.11}{1009.25} = 2.8056.$$

