Manual for SOA Exam FM/CAS Exam 2.

Chapter 6. Variable interest rates and portfolio insurance.

Section 6.5. Asset–liability management.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Hedging

The money collected by insurance companies is invested and subject to risk (possible losses). For example, bonds can drop in value if the rate of interest changes. Several methods have being developed to minimize the risk of investing. A method, often sophisticated, employed to minimize investment risk is called **hedging**. In this section, we study several hedging methods.

Match assets and liabilities

If possible we would like to match assets and liabilities. i.e. the total amount of contributions in assets equals the total amount of contributions in liabilities.

- If an insurance company has more liabilities than assets, it may fail to meet its commitments to its policyholders.
- If an insurance company has more assets than liabilities, it will not the make the appropriate profit for the capital available.

Example 1

A company has liabilities of 2000, 5000 and 10000 payable at the end of years 1, 2 and 5 respectively. The investments available to the company are the following zero–coupon 1000 par value bonds:

Bond	Maturity (years)	Effective Annual Yield
Bond A	1 year	4.5%
Bond B	2 years	5.0
Bond C	3 years	5.5
Bond D	4 years	6.0
Bond E	5 years	6.5

Determine the cost for matching these liabilities exactly.

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Solution: We need to buy 2 Bonds A, 5 Bonds B and 10 Bonds E. The cost is

$$2000(1.045)^{-1} + 5000(1.05)^{-2} + 10000(1.065)^{-5} = 13747.83136.$$

Example 2

A bond portfolio manager in a pension fund is designing a bond portfolio. His company has an obligation to pay 50000 at the end of each year for 3 years. He can purchase a combination of the following three bonds in order to exactly match its obligation:
(i) 1–year 5% annual coupon bond with a yield rate of 6%.
(ii) 2–year 7% annual coupon bond with a yield rate of 7%.
(ii) 3–year 9% annual coupon bond with a yield rate of 8%.
(i) How much of each bond should you purchase in order to exactly match the liabilities?
(ii) Find the cost of such a combination of bonds.

Solution: (i) Suppose that the face values of the bonds we buy are x, y, z, in each of the bonds, respectively. Then, the cashflows are

Liabilities	50000	50000	50000
Bond 1	$(1.05)x$	0	0
Bond 2	$(0.07)y$	$(1.07)y$	0
Bond 3	$(0.09)z$	$(0.09)z$	$(1.09)z$
Time	1	2	3

In order to match assets and liabilities,

$$\begin{aligned}50000 &= (1.05)x + (0.07)y + (0.09)z,\\ 50000 &= (1.07)y + (0.09)z,\\ 50000 &= (1.09)z.\end{aligned}$$

Hence,

$$\begin{aligned}z &= \frac{50000}{1.09} = 45871.56,\\ y &= \frac{50000-(0.09)(45871.56)}{1.07} = 42870.61645,\\ x &= \frac{50000-(0.07)(42870.61645)-(0.09)(45871.56)}{1.05} = 40829.15852.\end{aligned}$$

(ii) Find the cost of such a combination of bonds.
Solution: (ii) The total price of the bonds is

$$\begin{aligned}&(40829.15852)(1.05)(1.06)^{-1}\\&+(42870.61645)((0.07)a_{\overline{2}|7\%}+(1.07)^{-2})\\&+(45871.56)((0.09)a_{\overline{3}|8\%}+(1.08)^{-3})\\=&40035.97426+42870.61645+47053.71459=129960.3053.\end{aligned}$$

Theory of immunization

Fluctuation in interest rates can cause losses to a financial institution. Suppose that a financial institution has a cashflow of assets and a cashflow of liabilities. The present values of these cashflows is sensitive to changes of interest rates. If interest rates fall, the present value of the cashflow of liabilities increases. If interest rates increase, the present value of the cashflow of assets decreases. To mitigate the risk associated with a change in the interest rates, Redington (1954) introduced the **theory of immunization**. Immunization is a hedging method against the risk associated with changes in interest rates.

According with the traditional immunization theory, a portfolio is immunized against fluctuations in interest rates if the 3 criteria are satisfied:

1. The present value of the assets must equal the present value of the liabilities.
2. The duration of the assets must equal the duration of the liabilities.
3. The convexity of the assets must be greater than that of the liabilities.

The first of the previous 3 conditions is an efficiency condition. The last conditions are imposed so that interest rate risk for the assets offsets the interest rate risk for the liabilities.

If the the immunization conditions are satisfied, then

$$P_A(i) = P_L(i), \nu_A = \nu_L, \bar{c}_A > \bar{c}_L.$$

The approximations to the present value of assets and liabilities are:

$$P_A(i+h) \approx P_A(i)\left(1 - \nu_A h + \frac{h^2}{2}\bar{c}_A\right)$$

and

$$P_L(i+h) \approx P_L(i)\left(1 - \nu_L h + \frac{h^2}{2}\bar{c}_L\right).$$

Hence

$$P_A(i+h) - P_L(i+h) = P_A(i)\frac{h^2}{2}(\bar{c} - \bar{c}_L) > 0.$$

For small variation in interest rates the previous approximations are accurate. Since high variations in interest rates in short periods of time are unlikely, it is possible to hedge against interest rate variations by immunizing periodically.

Example 3

An actuarial department needs to set–up an investment program to pay for a loan of $20000 due in 2 years. The only available investments are:
(i) a money market fund paying the current rate of interest.
(ii) 5–year zero–coupon bonds earning 4%.
Assume that the current rate of interest is 4%. Develop an investment program satisfying the theory of immunization. Graph the present value of asset minus liabilities versus interest rates.

Solution: The investment program invest x in the money maker fund, and y in the zero coupon bond. The PV of the cashflow of assets and liabilities is
$P(i) = x + y(1.04)^5(1+i)^{-5} - 20000(1+i)^{-2}$. We solve for x and y such that $P(0.04) = 0$ and $P'(0.04) = 0$. Since
$P'(i) = -(5)y(1.04)^5(1+i)^{-6} + (2)(20000)(1+i)^{-3}$, we need to solve

$$x+y = 20000(1.04)^{-2}, \text{ and } -(5)y(1.04)^{-1}+(2)(20000)(1.04)^{-3} = 0.$$

We get $y = \frac{(2)(20000)(1.04)^{-2}}{5} = 7396.45$ and
$x = 20000(1.04)^{-2} - y = 11094.67$. Since

$$P''(i) = (5)(6)y(1.04)^5(1+i)^{-7} - (2)(3)(20000)(1+i)^{-4},$$

$$P''(0.04) = (5)(6)7396.45(1.04)^{-2}-(2)(3)(20000)(1.04)^{-4} = 102576.5.$$

The convexity of the cashflow is positive. The investment strategy consisting in allocate 11094.67 in the money market account and 7396.45 in bonds satisfies the immunization requirements.

The graph of the present value of asset minus liabilities versus interest rates is Figure 1.

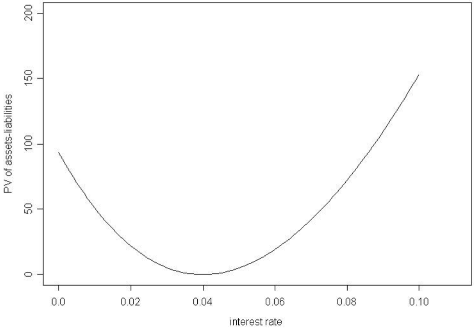

Figure 1: Present value of assets minus liabilities

Example 4

An actuarial department has determined that the company has a liability of $10,000 that will be payable in seven years. The company has two choices of assets to invest in: a 5–year zero-coupon bond and a 10–year zero coupon bond. The interest rate is 5%. How can the actuarial department immunize its portfolio?

Solution: The investment program invest x in the 5–year bond, and y in the 10–year bond. The PV of the cashflow of assets and liabilities is

$$P(i) = x(1.05)^5(1+i)^{-5} + y(1.05)^{10}(1+i)^{-10} - 10000(1+i)^{-7}.$$

So,

$$P'(i) = -(5)x(1.05)^5(1+i)^{-6} - (10)y(1.05)^{10}(1+i)^{-11} + (7)10000(1+i)^{-8}$$

and

$$P''(i) = (5)(6)x(1.05)^5(1+i)^{-7} + (10)(11)y(1.05)^{10}(1+i)^{-12} - (7)(8)10000(1+i)^{-9}.$$

We solve for x and y in the equations $P(0.05) = 0$ and $P'(0.05) = 0$, i.e.

$$x + y = (10000)(1.05)^{-7}, \text{ and } 5x + 10y = (70000)(1.05)^{-7}.$$

We get $x = (6000)(1.05)^{-7} = 4264.08$ and $y = (4000)(1.05)^{-7} = 2842.72$.

We have that

$$
\begin{aligned}
&P''(0.05) = (5)(6)(6000)(1.05)^{-7}(1.05)^{5}(1.05)^{-7}\\
&+ (10)(11)(4000)(1.05)^{-7}(1.05)^{10}(1.05)^{-12} - (7)(8)(10000)(1.05)^{-9}\\
=&(60000)(1.05)^{-9} > 0.
\end{aligned}
$$

So, the investment program satisfies the immunization requirements.

Example 5

An actuarial department has determined that the company has a liability of \$10,000 that will be payable in two years. The company has two choices of assets to invest in: a one–year zero–coupon bond and a three–year zero–coupon bond. The interest rate is 6%.
(i) Find an investment portfolio which immunizes this portfolio.
(ii) Find the interval of interest rates at which the present value of assets is bigger than the present value of liabilities.

Solution: (i) The investment program invest x in the one–year zero–coupon bond, and y in the three–year zero–coupon bond. The PV of the cashflow of assets and liabilities is

$$P(i) = x(1.06)(1+i)^{-1} + y(1.06)^3(1+i)^{-3} - 10000(1+i)^{-2}.$$

So,

$$P'(i) = x(1.06)(-1)(1+i)^{-2} + y(1.06)^3(-3)(1+i)^{-4} - 10000(-2)(1+i)^{-3}$$

and

$$P''(i) = x(1.06)(-1)(-2)(1+i)^{-3} + y(1.06)^3(-3)(-4)(1+i)^{-5} - 10000(-2)(-3)(1+i)^{-4}.$$

We solve for x and y in the equations $P(0.06) = 0$ and $P'(0.06) = 0$, i.e.

$$x + y = (10000)(1.06)^{-2}, \text{ and } x + 3y = (20000)(1.06)^{-2}.$$

We get $x = y = (5000)(1.06)^{-2} = 4449.98222.$

We have that

$$P''(0.06) = (10000)(1.06)^{-2} > 0.$$

So, the investment program satisfies the immunization requirements.
(ii) The present value of assets is bigger than the present value of liabilities if

$$0 \leq (5000)(1.06)^{-1}(1+i)^{-1} + (5000)(1.06)(1+i)^{-3} - 10000(1+i)^{-2}$$

which is equivalent to

$$\begin{aligned}
&(5000)(1.06)^{-1}(1+i)^{2} + (5000)(1.06) - 10000(1+i) \\
=&(5000)(1.06)^{-1}\left((1+i)^{2} - (2)(1.06) + (1.06)^{2}\right) \\
=&(5000)(1.06)^{-1}(i-0.06)^{2} \geq 0.
\end{aligned}$$

The present value of assets is bigger than the present value of liabilities for any interest rate.

