Manual for SOA Exam FM/CAS Exam 2.

Chapter 7. Derivatives markets.
Section 7.2. Forwards.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Forwards

Definition 1

A **forward** *is a contract between a buyer and seller in which they agree upon the sale of an asset of a specified quality for a specified price at a specified future date.*

Forward contracts are privately negotiated and are not standardized.

Common forwards are in commodities, currency exchange, stock shares and stock indices.

A forward contract states the following:

A forward contract states the following:

- forces the seller to sell and the buyer to buy.

A forward contract states the following:

- forces the seller to sell and the buyer to buy.
- spells out the quantity, quality and exact type of asset to be sold.

A forward contract states the following:

- forces the seller to sell and the buyer to buy.
- spells out the quantity, quality and exact type of asset to be sold.
- states the delivery price and the time, date and place for the transfer of ownership of the asset.

A forward contract states the following:

- forces the seller to sell and the buyer to buy.
- spells out the quantity, quality and exact type of asset to be sold.
- states the delivery price and the time, date and place for the transfer of ownership of the asset.
- specify the time, date, place for payment.

A forward contract states the following:

- forces the seller to sell and the buyer to buy.
- spells out the quantity, quality and exact type of asset to be sold.
- states the delivery price and the time, date and place for the transfer of ownership of the asset.
- specify the time, date, place for payment.

Usually a forward contract has more terms.

A forward contract states the following:

- forces the seller to sell and the buyer to buy.
- spells out the quantity, quality and exact type of asset to be sold.
- states the delivery price and the time, date and place for the transfer of ownership of the asset.
- specify the time, date, place for payment.

Usually a forward contract has more terms.
Sometimes instead of the asset to be delivered, there exists a cash settlement between the parties engaging in a forward contract.
Either **physical settlement** or **cash settlement** can be used to settle a forward contract.

A forward contract states the following:

- forces the seller to sell and the buyer to buy.
- spells out the quantity, quality and exact type of asset to be sold.
- states the delivery price and the time, date and place for the transfer of ownership of the asset.
- specify the time, date, place for payment.

Usually a forward contract has more terms.
Sometimes instead of the asset to be delivered, there exists a cash settlement between the parties engaging in a forward contract.
Either **physical settlement** or **cash settlement** can be used to settle a forward contract.
When entering in a forward contract, parties must check counterparts for credit risk (sometimes using a collateral, bank letters, real state guarantee, etc).

- The asset in which the forward contract is based is called the **underlier** or **underlying asset**.

- The asset in which the forward contract is based is called the **underlier** or **underlying asset**.
- The **nominal amount** (also called **notional amount**) of a forward contract is the quantity of the asset traded in the forward contract.

- The asset in which the forward contract is based is called the **underlier** or **underlying asset**.
- The **nominal amount** (also called **notional amount**) of a forward contract is the quantity of the asset traded in the forward contract.
- The price of the asset in the forward contract is called the **forward price**.

- The asset in which the forward contract is based is called the **underlier** or **underlying asset**.
- The **nominal amount** (also called **notional amount**) of a forward contract is the quantity of the asset traded in the forward contract.
- The price of the asset in the forward contract is called the **forward price**.
- The time at which the contract settles is called the **expiration date**.

For example, if a forward contract involves 10,000 barrels of oil to be delivered in one year, oil is the underlying asset, 10000 barrels is the notional amount and one year is the expiration date.

The two main reasons why an investor might be interested in forward contracts are: speculation and (hedging) reduce investment risk.

Apart from commission, a forward contract requires no initial payment. The current price of an asset is called its **spot price**. Besides the spot price, to price a forward contract, several factors, such as delivery cost and time of delivery must be taken into account.
The difference between the spot and the forward price is called the **forward premium** or **forward discount**.
If the forward price is higher than the spot price, the asset is **forwarded at a premium**. The premium is the forward price minus the current spot price.
If the forward price is lower than the spot price, the asset is **forwarded at a discount**. The discount is the current spot price minus the forward price.

The buyer of the asset in a forward contract is called the **long forward**. The long forward benefits when prices rise.
The seller of the asset in a forward contract is called the **short forward**. The short forward benefits when prices decline.

We will denote by S_T the spot price of an asset at time T. S_0 is the current price of the asset. S_0 is a fixed quantity. For $T > 0$, S_T is a random variable. We will denote by $F_{0,T}$ to the price at time zero of a forward with expiration time T paid at time T.
The **payoff** of a derivative is the value of this position at expiration.
The payoff of a long forward contract is $S_T - F_{0,T}$. Notice that the bearer of a long forward contract buys an asset at time T with value S_T for $F_{0,T}$. Assuming that there are no expenses setting the forward contract, the **profit** for a long forward is $S_T - F_{0,T}$.
The payoff of a short forward is $F_{0,T} - S_T$. The holder of a short forward contract sells at time T an asset with value S_T for $F_{0,T}$. Assuming that there are no expenses setting the forward contract, the profit for a short forward contract is $F_{0,T} - S_T$.

The profits of the long and the short in a forward contract are the opposite of each other. The sum of their profits is zero. A forward contract is a zero–sum game.

- The minimum long forward's profit $F_{0,T}$, which is attained when $S_T = 0$.
- The maximum long forward's profit is infinity, which is attained when $S_T = \infty$.
- The minimum short forward's profit is $-\infty$, which is attained when $S_T = \infty$.
- The maximum short forward's profit is $F_{0,T}$, which is attained when $S_T = 0$.

long forward's profit	$=$	$S_T - F_{0,T}$
short forward's profit	$=$	$F_{0,T} - S_T$

	minimum profit	maximum profit
long forward	$-F_{0,T}$	∞
short forward	$-\infty$	$-F_{0,T}$

Example 1

A gold miner enters a forward contract with a jeweler to sell him 200 ounces of gold in six months for $600 per ounce.
(i) Find the jeweler's payoff in the forward contract if the spot price at expiration of a gold ounce is $590, $595, $600, $605, $610. Graph the jeweler's payoff.
(ii) Find the gold miner's payoff in the forward contract if the spot price at expiration of a gold ounce is $590, $595, $600, $605, $610. Graph the gold miner's payoff.

Solution: (i) The jeweler's payoff is $(200)(S_T - 600)$. A table of the jeweler's payoff is

S_T	590	595	600	605	610
Payoff	-2000	-1000	0	1000	2000

The graph of the jeweler's payoff is given in Figure 1.
(ii) The gold miner's payoff is $(200)(600 - S_T)$. A table of the gold miner's payoff is

S_T	590	595	600	605	610
Payoff	2000	1000	0	-1000	-2000

Figure 1: Example 1. (Long forward) Jeweler's payoff

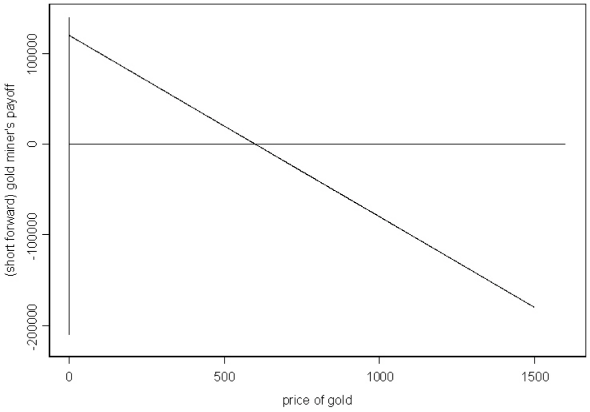

Figure 2: Example 1. (Short forward) Gold miner's payoff

Notice how figures 1 and 2 are the opposite of each other. The gold miner's payoff equals minus the jeweler's payoff.

Usually, a commissions has to be paid to enter a forward contract. Suppose that the long forward has to paid C_L to the market–maker at negotiation time to enter into the forward contract. Then, the profit for a long forward is

$$\text{Profit of a long forward} = S_T - F_{0,T} - C_L(1+i)^T,$$

where i is the annual effective rate of interest.
If the short forward has to paid C_S at negotiation time to the market–maker to enter the forward contract, then profit for a short forward is

$$\text{Profit of a short forward} = F_{0,T} - S_T - C_S(1+i)^T.$$

Sometimes, instead of using the annual effective rate of interest, we will use the **annual interest rate compounded continuously**. This is another name for the force of interest. This rate is also called the **annual continuous interest rate**. If r is the annual continuously compounded interest rate, then the future value at time T of a payment of P made at time zero is Pe^{rT}.

Alternative ways to buy an asset.

Suppose that you want to buy an asset. Suppose that the buyer's payment can be made at either time zero or time T. Suppose that the transfer of ownership of an asset can be made either at time zero or at time T. There are four possible ways to buy an asset (see Table 1):

Receive the asset at time	Pay at time	
	0	T
0	Outright purchase	Fully leveraged purchase
T	Prepaid forward contract	Forward contract

Table 1: Alternative ways to buy an asset.

1. **Outright purchase.** Both the payment and the transfer of ownership are made at time zero. The price paid per share is the current spot price S_0.
2. **Fully leveraged purchase.** The transfer of ownership is made at time zero. The payment is made at time T. The payment is $S_0 e^{rT}$, where S_0 is the current spot price and r is the risk–free continuously compounded annual interest rate.
3. **Prepaid forward contract.** A payment of $F_{0,T}^P$ is made at time zero. The transfer of ownership is made at time T. The payment $F_{0,T}^P$ is not necessarily the current spot price S_0.
4. **Forward contract.** Both the payment and the transfer of ownership happen at time T. The price of a forward contract is denoted by $F_{0,T}$. We have that $F_{0,T} = e^{rT} F_{0,T}^P$, where r is the risk–free annual interest rate continuously compounded.

Pricing a forward contract.

Whenever an asset is delivered and paid at time zero, the fair price of the asset is its (spot) market price. The market of an outright purchase is S_0. Commissions and bid–ask spreads must be taken into account.

The case of a fully leverage purchase, the price of a forward contract is just the price of a loan of S_0. The price of a fully leveraged purchase is $S_0 e^{rT}$, which is the price of a loan of S_0 taken at time zero and paid at time T.

We are interested in determining $F_{0,T}$, the price of a forward contract. Many different factors such as the cost of storing, delivering, the convenience yield and the scarcity of the asset. Some commodities like oil have high storage costs. The convenience yield measures the cost of not having the asset, but a forward contract on it. For example, if instead of having a forward on gasoline, we have the physical asset, we may use it in case of scarcity. In the case of stock paying dividends, an stock owner receives dividend payments, and a long forward holder does not.

Pricing a prepaid forward contract.

To find $F^P_{0,T}$, we make three cases.

1. Price of prepaid forward contract if there are no dividends. We consider an asset with no cost/benefit in holding the asset. This applies to the the price of a stock which does pay any dividends. It is irrelevant whether the transfer of ownership happens now or later
The no arbitrage price of a prepaid forward contract is $F^P_{0,T} = S_0$, where S_0 is the price of an asset today.

Example 2

XYZ stock costs $55 per share. XYZ stock does not pay any dividends. The risk–free interest rate continuously compounded 8%. Calculate the price of a prepaid forward contract that expires 30 months from today.

Example 2

XYZ stock costs $55 per share. XYZ stock does not pay any dividends. The risk–free interest rate continuously compounded 8%. Calculate the price of a prepaid forward contract that expires 30 months from today.

Solution: The prepaid forward price is $F_{0,T}^{P} = S_0 = 55$.

2. Price of prepaid forward contract when there are discrete dividends. Suppose that the stock is expected to make a dividend payment of D_{T_i} at the time t_i, $i = 1 \ldots, n$. A prepaid forward contract will entitle you receive the stock at time T without receiving the interim dividends. The prepaid forward price is

$$F_{0,T}^{P} = S_0 - \sum_{i=1}^{n} D_{t_i} e^{-rt_i}.$$

Example 3

XYZ stock cost \$55 per share. It pays \$2 in dividends every 3 months. The first dividend is paid in 3 months. The risk–free interest rate continuously compounded 8%. Calculate the price of a prepaid forward contract that expires 18 months from today, immediately after the dividend is paid.

Example 3

XYZ stock cost \$55 per share. It pays \$2 in dividends every 3 months. The first dividend is paid in 3 months. The risk–free interest rate continuously compounded 8%. Calculate the price of a prepaid forward contract that expires 18 months from today, immediately after the dividend is paid.

Solution: The prepaid forward price is

$$F_{0,T}^{P} = S_0 - \sum_{i=1}^{n} D_{t_i} e^{-rt_i} = 55 - \sum_{j=1}^{6} 2e^{-(0.08)j(1/4)}$$

$$=55 - \sum_{j=1}^{6} 2e^{-(0.02)j} = 55 - 2a_{\overline{6}|e^{0.02}-1} = 43.80474631.$$

Recall that $a_{\overline{n}|i} = \sum_{j=1}^{n}(1+i)^{-j}$.

3. Price of prepaid forward contract when there are continuous dividends. In the case of an index stock, dividends are given almost daily. We may model the dividend payments as a continuous flow. Let δ be the rate of dividends given per unit of time. Suppose that dividends payments are reinvested into stock. Let $t_j = \frac{jT}{n}$, $1 \le j \le n$. If A_j is the amount of shares at time t_j, then $A_{j+1} = A_j(1 + \frac{\delta T}{n})$. Hence, the total amount of shares multiplies by $1 + \frac{\delta T}{n}$ in each period. Hence, one share at time zero grows to $\left(1 + \frac{\delta T}{n}\right)^n$ at time T. Letting $n \to \infty$, we get that one share at time zero grows to $e^{\delta T}$ shares at time T. With \$$K$ at time 0, we can buy $\frac{K}{S_0}$ shares in the market at time 0. These shares grow to $\frac{K}{S_0}e^{\delta T}$ at time T. With \$$K$ at time 0, we can buy $\frac{K}{F^P_{0,T}}$ shares to be delivered at time T using a prepaid forward. Hence, is there exists no arbitrage, $\frac{K}{S_0}e^{T\delta} = \frac{K}{F^P_{0,T}}$ and

$$F^P_{0,T} = S_0 e^{-\delta T}.$$

Example 4

An investor is interested in buying XYZ stock. The current price of stock is \$45 per share. This stock pays dividends at an annual continuous rate of 5%. Calculate the price of a prepaid forward contract which expires in 18 months.

Example 4

An investor is interested in buying XYZ stock. The current price of stock is $45 per share. This stock pays dividends at an annual continuous rate of 5%. Calculate the price of a prepaid forward contract which expires in 18 months.

Solution: The price of the prepaid forward contract is

$$F_{0,T}^{P} = S_0 e^{-\delta T} = 45 e^{-(0.05)(18/12)} = 41.74845688.$$

Example 5

XYZ stock costs \$55 per share. The annual continuous interest rate is 0.055*. This stock pays dividends at an annual continuous rate of 3.5%. A one year prepaid forward has a price of \$52.60. Is there any arbitrage opportunity? If so, describe the position an arbitrageur would take and his profit per share.*

Example 5

XYZ stock costs \$55 per share. The annual continuous interest rate is 0.055. *This stock pays dividends at an annual continuous rate of 3.5%. A one year prepaid forward has a price of \$52.60. Is there any arbitrage opportunity? If so, describe the position an arbitrageur would take and his profit per share.*

Solution: The no arbitrage prepaid forward price is

$$F_{0,T}^P = S_0 e^{-\delta T} = 55e^{-0.035} = 53.10829789.$$

An arbitrage portfolio consists of entering a prepaid long forward contract for one share of stock and shorting $e^{-0.035}$ shares of stock. The return of this transaction is $55e^{-0.035} - 52.60 = 0.5082978942$. At redemption time, the arbitrageur covers his short position after executing the prepaid forward contract.

Example 6

XYZ stock costs $55 per share. The annual continuous interest rate is 0.035*. This stock pays dividends at an annual continuous rate of 5.5%. A one year prepaid forward has a price of $52.60. Is there any arbitrage opportunity? If so, describe the position an arbitrageur would take and his profit per share.*

Example 6

XYZ stock costs $55 per share. The annual continuous interest rate is 0.035*. This stock pays dividends at an annual continuous rate of 5.5%. A one year prepaid forward has a price of $52.60. Is there any arbitrage opportunity? If so, describe the position an arbitrageur would take and his profit per share.*

Solution: The no arbitrage prepaid forward price is

$$F_{0,T}^{P} = S_0 e^{-\delta T} = 55e^{-0.055} = 52.05668314.$$

An arbitrage portfolio consists of entering a prepaid short forward contract for one share of stock and buying $e^{-0.055}$ shares of stock. The return of this transaction is $52.60 - 55e^{-0.055} = 0.5433168626$. At redemption time, we use the bought stock to meet the short forward.

Notice that in the previous questions, we can make arbitrage without making any investment of capital. The total price of setting the portfolios at time zero is zero.

Pricing a forward contract.

Both the payment and the transfer of ownership happen at time T. The price of a forward contract is the future value of the prepaid forward contract, i.e. $F_{0,T} = e^{rT} F^{P}_{0,T}$. So,

- The price of a forward contract for a stock with no dividends is $F_{0,T} = e^{rT} S_0$.
- The price of a forward contract for a stock with discrete dividends is $F_{0,T} = e^{rT} S_0 - \sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)}$.
- The price of a forward contract for a stock with continuous dividends is $F_{0,T} = e^{(r-\delta)T} S_0$.

Example 7

The current price of one share of XYZ stock is 55.34. The price of a nine–month forward contract on one share of XYZ stock is 57.6. XYZ stock is not going to pay any dividends on the next 2 years.
(i) Calculate the annual compounded continuously interest rate implied by this forward contract.
(ii) Calculate the price of a two–year forward contract on one share of XYZ stock.

Example 7

The current price of one share of XYZ stock is 55.34. The price of a nine–month forward contract on one share of XYZ stock is 57.6. XYZ stock is not going to pay any dividends on the next 2 years.
(i) Calculate the annual compounded continuously interest rate implied by this forward contract.
(ii) Calculate the price of a two–year forward contract on one share of XYZ stock.

Solution: (i) Since $F_{0,T} = e^{rT} S_0$, $57.6 = e^{(3/4)r} 55.34$ and $r = (4/3) \ln(57.6/55.34) = 0.05336879112$.

Example 7

The current price of one share of XYZ stock is 55.34. The price of a nine–month forward contract on one share of XYZ stock is 57.6. XYZ stock is not going to pay any dividends on the next 2 years.
(i) Calculate the annual compounded continuously interest rate implied by this forward contract.
(ii) Calculate the price of a two–year forward contract on one share of XYZ stock.

Solution: (i) Since $F_{0,T} = e^{rT} S_0$, $57.6 = e^{(3/4)r} 55.34$ and $r = (4/3)\ln(57.6/55.34) = 0.05336879112$.
(ii) We have that
$F_{0,2} = e^{r2} S_0 = e^{(0.05336879112)(2)} 55.34 = 61.57362151$.

Example 8

A stock is expected to pay a dividend of $1 per share in 2 months and again in 5 months. The current stock price is $59 per share. The risk free effective annual rate of interest is 6%.
(i) What is the fair price of a 6–month forward contract?
(ii) Assume that 3 months from now the stock price is $57 per share, what is the fair price of the same forward contract at that time?

Example 8

A stock is expected to pay a dividend of \$1 per share in 2 months and again in 5 months. The current stock price is \$59 per share. The risk free effective annual rate of interest is 6%.
(i) What is the fair price of a 6–month forward contract?
(ii) Assume that 3 months from now the stock price is \$57 per share, what is the fair price of the same forward contract at that time?

Solution: (i) The forward price is the future value of the payments associated with owning the stock in six months:
$F_{0,0.5} = (59)(1.06)^{0.5} - (1)(1.06)^{4/12} - (1)(1.06)^{1/12} = 58.71974.$

Example 8

A stock is expected to pay a dividend of $1 per share in 2 months and again in 5 months. The current stock price is $59 per share. The risk free effective annual rate of interest is 6%.
(i) What is the fair price of a 6–month forward contract?
(ii) Assume that 3 months from now the stock price is $57 per share, what is the fair price of the same forward contract at that time?

Solution: (i) The forward price is the future value of the payments associated with owning the stock in six months:
$F_{0,0.5} = (59)(1.06)^{0.5} - (1)(1.06)^{4/12} - (1)(1.06)^{1/12} = 58.71974.$
(ii) $(57)(1.06)^{3/12} - (1)(1.06)^{1/12} = 56.83154.$

Example 9

An investor is interested in buying XYZ stock. The current price of stock is \$30 per share. The risk–free annual interest rate continuously compounded is 0.03*. The price of a fourteen–month forward contract is* 30.352*. Calculate the continuous dividend yield* δ*.*

Example 9

An investor is interested in buying XYZ stock. The current price of stock is $30 per share. The risk–free annual interest rate continuously compounded is 0.03. The price of a fourteen–month forward contract is 30.352. Calculate the continuous dividend yield δ.

Solution: We have that

$$30.352 = F_{0,T} = S_0 e^{(r-\delta)T} = 30e^{(0.03-\delta)(14/12)}.$$

and

$$\delta = 0.03 - (12/14)\ln(30.352/30) = 0.02000140155.$$

Example 10

An investor is interested in buying XYZ stock. The current price of stock is \$30 per share. This stock pays dividends at an annual continuous rate of 0.02. The risk–free annual effective rate of interest is 0.045.
(i) What is the price of prepaid forward contract which expires in 18 months?
(ii) What is the price of forward contract which expires in 18 months?

Example 10

An investor is interested in buying XYZ stock. The current price of stock is \$30 per share. This stock pays dividends at an annual continuous rate of 0.02. The risk–free annual effective rate of interest is 0.045.
(i) What is the price of prepaid forward contract which expires in 18 months?
(ii) What is the price of forward contract which expires in 18 months?

Solution: (i) The prepaid forward price is

$$F_{0,T}^{P} = S_0 e^{-\delta T} = 30e^{-(0.02)(18/12)} = 29.11336601.$$

Example 10

An investor is interested in buying XYZ stock. The current price of stock is $30 per share. This stock pays dividends at an annual continuous rate of 0.02. The risk–free annual effective rate of interest is 0.045.
(i) What is the price of prepaid forward contract which expires in 18 months?
(ii) What is the price of forward contract which expires in 18 months?

Solution: (i) The prepaid forward price is

$$F_{0,T}^{P} = S_0 e^{-\delta T} = 30e^{-(0.02)(18/12)} = 29.11336601.$$

(ii) The 18–month forward price is
$29.11336601(1.045)^{18/12} = 31.1004631.$

off–market forward contract

A forward contract where either you pay a premium or you collect a premium for entering into the deal is called an **off–market forward contract**.

Example 11

Suppose that the current value of a certain amount of a commodity is \$45000. The annual effective rate of interest is 4.5%.
(i) You are offered a 2–year long forward contract at a forward price of \$50000. How much would you need to be paid to enter into this contract?
(ii) You are offered a 2–year long forward contract at a forward price of \$48000. How much would you need be willing to pay to enter into this contract?

Example 11

Suppose that the current value of a certain amount of a commodity is \$45000. The annual effective rate of interest is 4.5%.
(i) You are offered a 2–year long forward contract at a forward price of \$50000. How much would you need to be paid to enter into this contract?
(ii) You are offered a 2–year long forward contract at a forward price of \$48000. How much would you need be willing to pay to enter into this contract?

Solution: (i) Let x be how much you need to be paid to enter into this contract. The current value of the commodity should be equal to the present value of the expenses needed to get the commodity using the long forward contract. Hence, $50000(1.045)^{-2} - x = 45000$. So, $x = 50000(1.045)^{-2} - 45000 = 786.4976$.

Example 11

Suppose that the current value of a certain amount of a commodity is \$45000. The annual effective rate of interest is 4.5%.
(i) You are offered a 2–year long forward contract at a forward price of \$50000. How much would you need to be paid to enter into this contract?
(ii) You are offered a 2–year long forward contract at a forward price of \$48000. How much would you need be willing to pay to enter into this contract?

Solution: (ii) Let y be how much would you need be willing to pay to enter into this contract. The current value of the commodity should be equal to the present value of the expenses needed to get the commodity using the long forward contract. Hence, $48000(1.045)^{-2} + y = 45000$. So, $y = 45000 - 48000(1.045)^{-2} = 1044.962341$.

Arbitrage

If the price of a forward contract does not follow the previous formulas, an arbitrageur can do arbitrage.

Example 12

XYZ stock pays no dividends and has a current price of \$42.5 per share. A long position in a forward contract is available to buy 1000 shares of stock six months from now for \$43 per share. A bank pays interest at the rate of 5% per annum (continuously compounded) on a 6–month certificate of deposit. Describe a strategy for creating an arbitrage profit and determine the amount of the profit.

Example 12

XYZ stock pays no dividends and has a current price of \$42.5 per share. A long position in a forward contract is available to buy 1000 shares of stock six months from now for \$43 per share. A bank pays interest at the rate of 5% per annum (continuously compounded) on a 6–month certificate of deposit. Describe a strategy for creating an arbitrage profit and determine the amount of the profit.

Solution: The no arbitrage price of a forward contract is $S_0 e^{rT} = (42.5)e^{0.05(0.5)} = 43.57589262$. Hence, it is possible to do arbitrage by entering into the long forward position. An arbitrageur can: sell 1000 shares of stock for $(1000)(42.5) = 42500$, deposit 42500 in the bank for six months, and sign up a forward contract for a long position for 1000 shares of stock. In six months, the CD returns $(1000)(42.5)e^{0.05(0.5)} = 43575.89262$. The cost of the forward is $(1000)(43) = 430000$. Hence, the profit is $43575.89262 - 430000 = 575.89262$.

Example 13

Suppose that the risk–free effective rate of interest is 5% per annum. XYZ stock is currently trading for $45.34 per share. XYZ stock is expected to pay a dividend of $1.20 per share six months from now. The price of a nine–month forward contract on one share of XYZ stock is $47.56. Is there an arbitrage opportunity on the forward contract? If so, describe the strategy to realize profit and find the arbitrage profit.

Example 13

Suppose that the risk–free effective rate of interest is 5% per annum. XYZ stock is currently trading for \$45.34 per share. XYZ stock is expected to pay a dividend of \$1.20 per share six months from now. The price of a nine–month forward contract on one share of XYZ stock is \$47.56. Is there an arbitrage opportunity on the forward contract? If so, describe the strategy to realize profit and find the arbitrage profit.

Solution: The no arbitrage forward price is

$$F_{0,T} = e^{rT} S_0 - \sum_{i=1}^{n} D_{t_i} e^{r(T-t_i)} = 45.34(1.05)^{9/12} - 1.2(1.05)^{3/12}$$

$$=45.81511211.$$

We can make arbitrage by buying stock and entering a short forward contract. The profit per share at expiration is $47.56 - 45.81511211 = 1.74488789$.

Suppose that a stock pays dividends at the continuous rate δ. In the absence of arbitrage, entering a forward for one share for $F_{0,T}$ is equivalent to buying $e^{-\delta T}$ shares of stock for $S_0e^{-\delta T}$ and (borrowing $S_0e^{-\delta T}$) selling a zero–coupon bond for $S_0e^{-\delta T}$ with expiration in T years. In both cases, at time T we have one share of stock after we make a payment of $F_{0,T}$. There exists no arbitrage if $S_0e^{-\delta T} = F_{0,T}e^{-rT}$. If $S_0e^{-\delta T} \neq F_{0,T}e^{-rT}$, we can make arbitrage.

If $F_{0,T} < e^{(r-\delta)T}S_0$, we can enter into a long forward for one share of stock, and short $e^{-\delta T}$ shares of stock. At redemption time, we cover the short position by paying $F_{0,T}$ for the stock. It is like we have borrowed $S_0e^{-\delta T}$ and pay the loan for $F_{0,T}$. In some sense we have created a zero–coupon bond. The position is called a **synthetic zero–coupon bond**. Let r' be the continuous annual rate of interest of the synthetic bond. This rate is called the **implied repo rate**. We have that

$$S_0e^{-\delta T}e^{r'T} = F_{0,T}.$$

Hence, if $F_{0,T} < e^{(r-\delta)T}S_0$,

$$r' = \frac{1}{T}\log\left(\frac{F_{0,T}}{S_0e^{-\delta T}}\right) < \frac{1}{T}\log\left(\frac{S_0e^{(r-\delta)T}}{S_0e^{-\delta T}}\right) < r.$$

By doing an arbitrage, we are able to reduce the interest rate at which we borrow. Technically, this is not call arbitrage. It is called **quasi–arbitrage**. We benefit from this portfolio, only if we are already borrowing.

Notice that the synthetic bond is created observing that

$$\text{Long forward} = \text{Buy stock} + \text{Issue a bond}$$

implies that

$$\text{Issue a bond} = \text{Short stock} + \text{Long forward}$$

Reciprocally, if $F_{0,T} > e^{(r-\delta)T}S_0$, we can create a portfolio earning a rate of interest bigger than the risk–free interest rate. We can enter a short forward contract for one share of stock and buy $e^{-\delta T}$ shares of stock. At redemption time, we get an inflow of $F_{0,T}$. Since we invested $S_0 e^{-\delta T}$, the continuous annual interest rate r', which we earned in the investment satisfies

$$S_0 e^{-\delta T} e^{r'T} = F_{0,T}.$$

Hence, if $F_{0,T} > e^{(r-\delta)T}S_0$,

$$r' = \frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0 e^{-\delta T}}\right) > \frac{1}{T} \ln\left(\frac{S_0 e^{(r-\delta)T}}{S_0 e^{-\delta T}}\right) = r.$$

Again this rate is called the implied repo rate.

Example 14

XYZ stock costs \$123.118 per share. This stock pays dividends at an annual continuous rate of 2.5%. A 18 month forward has a price of \$130.242. You own 10000 shares of XYZ stock. Calculate the annual continuous rate of interest at which you can borrow by shorting your stock.

Example 14

XYZ stock costs \$123.118 per share. This stock pays dividends at an annual continuous rate of 2.5%. A 18 month forward has a price of \$130.242. You own 10000 shares of XYZ stock. Calculate the annual continuous rate of interest at which you can borrow by shorting your stock.

Solution: We have that

$$r' = \delta + \frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0}\right) = 0.025 + \frac{1}{1.5} \ln\left(\frac{130.242}{123.118}\right) = 6.250067554\%.$$

Example 15

XYZ stock costs \$124 per share. This stock pays dividends at an annual continuous rate of 1.5%. A 2–year forward has a price of \$135.7 per share. Calculate the annual continuous rate of interest which you earn by buying stock and entering into a short forward contract, both positions for the same nominal amount.

Example 15

XYZ stock costs $124 per share. This stock pays dividends at an annual continuous rate of 1.5%. A 2–year forward has a price of $135.7 per share. Calculate the annual continuous rate of interest which you earn by buying stock and entering into a short forward contract, both positions for the same nominal amount.

Solution: We have that

$$r' = \delta + \frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0}\right) = 0.015 + \frac{1}{2} \ln\left(\frac{135.7}{124}\right) = 6\%.$$

The **forward premium** is $\frac{F_{0,T}}{S_0}$. Notice that this is not a price. Prices of options are called premiums. But, here the nomenclature is different. If a stock index pays dividends according with a continuous rate δ, then $\frac{F_{0,T}}{S_0} = e^{T(r-\delta)}$.

Example 16

XYZ stock cost \$55 per share. A four–month forward on XYZ stock costs \$57.5. XYZ stock pays dividends according a continuous rate.

(i) Calculate the four–month forward premium.

(ii) Calculate the eight–month forward premium.

(iii) Calculate the eight–month forward price.

Example 16

XYZ stock cost \$55 per share. A four–month forward on XYZ stock costs \$57.5. XYZ stock pays dividends according a continuous rate.
(i) Calculate the four–month forward premium.
(ii) Calculate the eight–month forward premium.
(iii) Calculate the eight–month forward price.

Solution: (i) The four–month forward premium is
$\frac{F_{0,4/12}}{S_0} = \frac{57.5}{55} = 1.045454545.$

Example 16

XYZ stock cost \$55 per share. A four–month forward on XYZ stock costs \$57.5. XYZ stock pays dividends according a continuous rate.
(i) Calculate the four–month forward premium.
(ii) Calculate the eight–month forward premium.
(iii) Calculate the eight–month forward price.

Solution: (i) The four–month forward premium is $\frac{F_{0,4/12}}{S_0} = \frac{57.5}{55} = 1.045454545$.
(ii) The eight–month forward premium is

$$\frac{F_{0,8/12}}{S_0} = e^{(8/12)(r-\delta)} = \left(e^{(4/12)(r-\delta)}\right)^2 = \left(\frac{57.5}{55}\right)^2 = 1.092975206.$$

Example 16

XYZ stock cost \$55 per share. A four–month forward on XYZ stock costs \$57.5. XYZ stock pays dividends according a continuous rate.
(i) Calculate the four–month forward premium.
(ii) Calculate the eight–month forward premium.
(iii) Calculate the eight–month forward price.

Solution: (i) The four–month forward premium is $\frac{F_{0,4/12}}{S_0} = \frac{57.5}{55} = 1.045454545$.
(ii) The eight–month forward premium is

$$\frac{F_{0,8/12}}{S_0} = e^{(8/12)(r-\delta)} = \left(e^{(4/12)(r-\delta)}\right)^2 = \left(\frac{57.5}{55}\right)^2 = 1.092975206.$$

(iii) The eight–month forward price is
$F_{0,8/12} = (55)(1.092975206) = 60.11363633$.

The **annualized forward premium** is $\frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right)$. Note that in the case of continuous dividends, the annualized forward premium is $r-\delta$.

Example 17

XYZ stock cost \$55 per share. A four–month forward on XYZ stock costs \$57.5.
(i) Calculate the annualized forward premium
(ii) Calculate the twelve–month forward price.

Example 17

XYZ stock cost $55 per share. A four–month forward on XYZ stock costs $57.5.
(i) Calculate the annualized forward premium
(ii) Calculate the twelve–month forward price.

Solution: (i) The annualized forward premium is

$$r - \delta = \frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0}\right) = \frac{1}{4/12} \ln\left(\frac{57.5}{55}\right) = 0.1333552877.$$

Example 17

XYZ stock cost \$55 per share. A four–month forward on XYZ stock costs \$57.5.
(i) Calculate the annualized forward premium
(ii) Calculate the twelve–month forward price.

Solution: (i) The annualized forward premium is

$$r-\delta=\frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right)=\frac{1}{4/12}\ln\left(\frac{57.5}{55}\right)=0.1333552877.$$

(ii) The twelve–month forward price is

$$F_{0,T}=S_0e^{r-\delta}=55e^{0.1333552877}=62.84607438.$$

Hedging a forward contract.

A (scalper) market maker must be able to offset the risk of trading forward contracts. Assume continuous dividends. Suppose that a scalper enters into a short forward contract. The profit at expiration for a long forward position is $S_T - F_{0,T}$. In order to obtain this same payoff a scalper can borrow $S_0 e^{-\delta T}$ and use this money to get $e^{-\delta T}$ shares of stock. At time T, he sells the stock which he owns for $S_0 e^{(r-\delta)T} = F_{0,T}$. Notice that by investing the dividends, $e^{-\delta T}$ shares of stock have grown to one share at time T. Borrowing $S_0 e^{-\delta T}$ and buying $e^{-\delta T}$ shares of stock is called a **synthetic long forward**. So, if a scalper enters into a short forward contract with a client, the scalper either matches this position with another client's long forward contract or creates a synthetic long forward

The profit of a short forward position is $F_{0,T} - S_T$. A scalper can get this profit, by (lending) buying a zero–coupon bond for $S_0 e^{-\delta T}$ and shorting $e^{-\delta T}$ shares receiving $S_0 e^{-\delta T}$. Buying a zero–coupon bond for $S_0 e^{-\delta T}$ and shorting a tailed position for $e^{-\delta T}$ shares is called a **synthetic short forward**. Again, a scalper may need to create this position to match a client's long forward position.

Using these strategies, a market–maker can hedge his clients positions.
A transaction in which you buy the asset and short the forward contract is called **cash–and–carry** (or **cash–and–carry hedge**). It is called cash–and–carry, because the cash is used to buy the asset and the asset is kept. A cash–and–carry has no risk. You have obligation to deliver the asset, but you also own the asset. An arbitrage that involves buying the asset and selling it forward is called **cash–and–carry arbitrage**. A (**reverse cash–and–carry hedge**) **reverse cash–and–carry** involves short–selling and asset and entering into a long forward position.

An arbitrageur can make money if $F_{0,T} \neq S_0 e^{(r-\delta)T}$. But, in the real world, transaction costs have to be taken into account.
Suppose that:
(i) The stock bid and ask prices are S_0^b and S_0^a, where $S_0^b < S_0^a$.
(ii) The forward bid and ask prices are $F_{0,T}^b < F_{0,T}^a$.
(iii) The cost of a transaction in the stock is K_S.
(iv) The cost of a transaction in the forward is K_F.
(v) The interest rates for borrowing and lending are $r_b > r_l$, respectively.

Suppose that the arbitrageur believes that the observed forward price $F_{0,T}$ is too high. Then, he could:
(a) contract a short forward for $F^{b}_{0,T}$
(b) buy a tailed position in stock for $S^{a}_{0}e^{-\delta T}$.
(c) borrow $S_0^{ae^{-\delta T}} + K_F + K_S$.
The payoff of this combined transaction is:

$$\begin{aligned}&F^{b}_{0,T} - S_T + S_T - (S^{a}_{0}e^{-\delta T} + K_f + K_S)e^{r_b T}\\=&F^{b}_{0,T} - (S^{a}_{0}e^{-\delta T} + K_f + K_S)e^{r_b T}.\end{aligned}$$

The scalper makes money if $F^{b}_{0,T} > (S^{a}_{0}e^{-\delta T} + K_f + K_S)e^{r_b T}$. The previous strategy is **cash–and–carry arbitrage**.

Suppose that the scalper believes that the observed forward price $F_{0,T}$ is too low. Then, he could:
(a) enter a long forward for $F_{0,T}^a$
(b) short a tailed position in stock for $S_0^b e^{-\delta T}$.
(c) lend $S_0^b e^{-\delta T} - K_F - K_S$.
The payoff of this combined transaction is:

$$\begin{aligned} & S_T - F_{0,T}^a - S_T + (S_0^a e^{-\delta T} - K_F - K_S)e^{r_l T} \\ = & - F_{0,T}^a + (S_0^a e^{-\delta T} - K_F - K_S)e^{r_l T}. \end{aligned}$$

The arbitrageur makes money if $(S_0^a e^{-\delta T} - K_F - K_S)e^{r_l T} > F_{0,T}^a$.
The previous strategy is **reverse cash–and–carry arbitrage**.

Example 18

Suppose that an arbitrageur would like to enter a cash–and–carry for 10000 barrels of oil for delivery in six months. Suppose that he can borrow at an annual effective rate of interest of 4.5%. The current price of a barrel of oil is \$55.
(i) What is the minimum forward price at which he would make a profit?
(ii) What is his profit if the forward price is \$57?

Example 18

Suppose that an arbitrageur would like to enter a cash–and–carry for 10000 barrels of oil for delivery in six months. Suppose that he can borrow at an annual effective rate of interest of 4.5%. The current price of a barrel of oil is $55.
(i) What is the minimum forward price at which he would make a profit?
(ii) What is his profit if the forward price is $57?

Solution: (i) He would make a profit if
$F_{0,T} > 55(1.045)^{1/2} = 56.22388283.$

Example 18

Suppose that an arbitrageur would like to enter a cash–and–carry for 10000 barrels of oil for delivery in six months. Suppose that he can borrow at an annual effective rate of interest of 4.5%. The current price of a barrel of oil is $55.

(i) What is the minimum forward price at which he would make a profit?

(ii) What is his profit if the forward price is $57?

Solution: (i) He would make a profit if $F_{0,T} > 55(1.045)^{1/2} = 56.22388283$.

(ii) The profit is $(10000)(57 - (55)(1.045)^{1/2}) = 7761.171743$.

