

# Manual for SOA Exam FM/CAS Exam 2.

Chapter 7. Derivatives markets.  
Section 7.9. Risk Management.

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# Risk Management

The main reasons why firms enter derivatives are: to hedge, to speculate, and to reduce transactions costs. Managers use derivatives taking in account their view of the market. So, a speculative component is added to each decision.

Firms convert inputs, such a labor and raw materials, into goods and services. A firm makes money if its income exceeds its costs. A change in the price of raw materials could make the firm unprofitable. A firm can use derivatives to alter its risk and protect its profitability. To do this is to do risk management.

Firms do risk management to attain **financial stability**. There are many reasons to avoid large losses. After a large loss, interest rates on loans will be obtained at a higher rate. Large losses can cause bankruptcy and distress costs. After a large loss, a company can face low cashflows and difficulty making fixed obligations such as wages and payments to banks and suppliers. This makes more costly to find employees, debtors and suppliers.

Suppose that the profit can be modeled by a random variable  $X$ . Let  $f(X)$  be the the profit after taking in account the effects of this profit. Because of the reasons before,  $f$  can be modeled using a concave function.

## Example 1

Suppose that the financial total impact of the profit is

$$f(x) = \begin{cases} 1.5x & \text{if } x \leq -2, \\ 1.1x & \text{if } -2 < x \leq 0 \\ x & \text{if } x > 0. \end{cases}$$

Since

$$f'(x) = \begin{cases} 1.5 & \text{if } x \leq -2, \\ 1.1 & \text{if } -2 < x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

is a nonincreasing function,  $f$  is a concave function.

## Example 2

Suppose that the a company profit before taxes can be modeled by a random variable  $X$ . The company pays 35% of its profit in taxes, if the profit is positive. It does not pay any taxes if its profit is negative. The company's profit after taxes is  $f(X)$ , where

$$f(x) = \begin{cases} (0.65)x & \text{if } x > 0, \\ x & \text{if } x \leq 0. \end{cases}$$

$f(x)$  is a concave function.

If the company is able to hedge and attain a constant profit of  $E[X]$ , then after taxes its profit is  $f(E[X])$ . If the company does not hedge its profit, its expected profit after taxes is  $E[f(X)]$ . Since  $f$  is a concave function, by the Jensen's inequality,

$$E[f(X)] \leq f(E[X]).$$

### Theorem 1

*(Jensen's inequality)* Let  $X$  be a random variable. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Then,

- (i) If  $f$  is convex, then  $f(E[X]) \leq E[f(X)]$ .
- (ii) If  $f$  is concave, then  $E[f(X)] \leq f(E[X])$ .

By using hedging, the random profit  $X$  is changed into a random profit with less variation (taking big losses with less probability). The expectation of this new random profit is larger than the expectation of the original profit.

### Example 3

*Hank is a wheat farmer. He will produce 50000 bushels of wheat at the end of one year. The cost of producing this wheat is \$5.7 per bushel. The price of wheat per bushel in one year will be:*

$S_1$	5.5	6.5
Probability	0.5	0.5

*A speculator offers short forward contracts for a price equal to 1% less than the expected value of the price of the wheat. It also offers a 6–strike put option with a price equal to 1% more than the present value of the expected payoff of this put. The annual effective rate of interest is 5%. Hank pays 30% of its profits on taxes and has no tax benefits for losses.*

(i) Calculate the prices of the forward contract and the put option.

(i) Calculate the prices of the forward contract and the put option.

**Solution:** (i) The expected price of wheat is

$$E[S_1] = (0.5)(5.5) + (0.5)(6.5) = 6.00/\text{bush}.$$

The price of a forward contract is  $6(0.99) = 5.94/\text{bush}$ .

The expected payoff of the put is

$$\begin{aligned} E[\max(6 - S_1, 0)] &= (0.5) \max(6 - 5.5, 0) + (0.5) \max(6 - 6.5, 0) \\ &= (0.5)(6 - 5.5) + (0.5)(0) = 0.25. \end{aligned}$$

The price of a put contract is

$$(0.25)(1.01)(1.05)^{-1} = 0.2404761905/\text{bush}.$$

(ii) Calculate Hank's expected profit before taxes if (a) he does not buy any derivative, (b) he enters a short forward contract, (c) he buys a put option. Which strategy has the biggest expected profit before taxes?

(ii) Calculate Hank's expected profit before taxes if (a) he does not buy any derivative, (b) he enters a short forward contract, (c) he buys a put option. Which strategy has the biggest expected profit before taxes?

**Solution:** (ii) For an uninsured position, Hank's profit before taxes is  $50000(S_1 - 5.7)$ . Hank's expected profit before taxes is

$$50000(E[S_1] - 5.7) = 50000(6 - 5.7) = 15000.$$

Entering the short forward contract, Hank's profit before taxes is

$$50000(S_1 - 5.7 + 5.94 - S_1) = (50000)(5.94 - 5.7) = 12000.$$

Entering the short forward contract, Hank's expected profit before taxes is 12000.

Buying the put option, Hank's profit before taxes is

$$\begin{aligned} & 50000 (S_1 - 5.7 + \max(6 - S_1, 0) - (0.2404761905)(1.05)) \\ &= 50000 (\max(6, S_1) - 5.7 - (0.2404761905)(1.05)) \\ &= 50000(\max(6, S_1) - 5.9525). \end{aligned}$$

Buying the put option, Hank's expected profit before taxes is

$$\begin{aligned} & E[50000 (\max(6, S_1) - 5.9525)] \\ &= 50000 (E[\max(6, S_1)] - 5.9525) \\ &= (50000)((0.5)(6) + (0.5)(6.5) - 5.9525) = 14875. \end{aligned}$$

The biggest expected profit before taxes is attained for an uninsured position.

(iii) Calculate Hank's expected profit after taxes for each of three strategies in (ii). Which strategy has the biggest expected profit after taxes?

(iii) Calculate Hank's expected profit after taxes for each of three strategies in (ii). Which strategy has the biggest expected profit after taxes?

**Solution:** (iii) For an uninsured position, Hank's profit before taxes is

$$50000(S_1 - 5.7) = \begin{cases} 50000(5.5 - 5.7) & \text{if } S_1 = 5.5, \\ 50000(6.5 - 5.7) & \text{if } S_1 = 6.5, \end{cases}$$

Hank's profit after taxes is

$$50000(S_1 - 5.7) = \begin{cases} 50000(5.5 - 5.7) & \text{if } S_1 = 5.5, \\ 50000(6.5 - 5.7)(0.7) & \text{if } S_1 = 6.5, \end{cases}$$

Hank's expected profit after taxes is

$$(0.5)(50000)(5.5 - 5.7) + (0.5)(50000)(6.5 - 5.7)(0.7) = 9000.$$

Entering the forward contract, Hank's expected profit after taxes is  $12000(0.7) = 8400$ .

Buying the put option, Hank's profit before taxes is

$$\begin{aligned}
 &= 50000 (\max(6, S_1) - 5.9525) \\
 &= \begin{cases} 50000(6 - 5.9525) & \text{if } S_1 = 5.5, \\ 50000(6.5 - 5.9525) & \text{if } S_1 = 6.5, \end{cases}
 \end{aligned}$$

Hank's profit after taxes is

$$\begin{aligned}
 &50000 (\max(6, S_1) - 5.9525) \\
 &= \begin{cases} 50000(6 - 5.9525)(0.7) & \text{if } S_1 = 5.5, \\ 50000(6.5 - 5.9525)(0.7) & \text{if } S_1 = 6.5, \end{cases}
 \end{aligned}$$

Hank's expected profit after taxes is

$$(0.5)(50000)(6 - 5.9525)(0.7) + (0.5)(50000)(6.5 - 5.9525)(0.7) = 10412.5$$

The biggest expected profit after taxes is attained when buying the put.

There are several reasons not to hedge:

- ▶ paying transaction costs.
- ▶ need expertise to assess costs and benefits of a given strategy.
- ▶ need expertise to do accounting and taxes in derivative transactions.

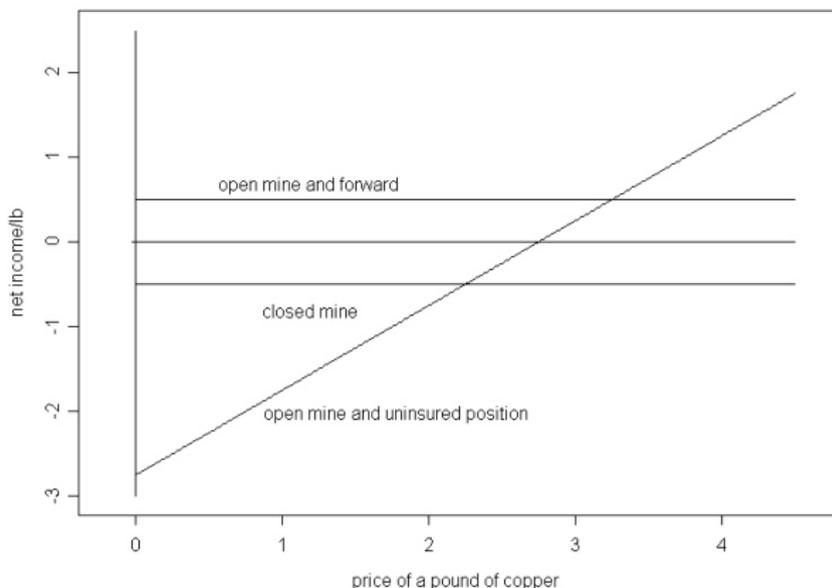
In the real world, small companies are discouraged to do derivatives because the reasons above. However, large companies have financial, accounting and legal departments which allow them to take advantage of the opportunities on market derivatives. The financial department of a large company can assess derivatives as well or better than the market does. Their legal and accounting departments allow them to take advantage of the current tax laws.

## A producer perspective on risk management.

(URMC) Utah Red Mountain Company is a copper–mining company. It plans to mine and sell 1,000,000 pounds of copper over the next year. Suppose that it sells all the next year's production, precisely one year from today. Suppose that the firm incurs in two types of costs: fixed costs and variable costs. Fixed costs are assumed whether its mine is open or closed. Variable costs are assumed only its mine is open. Suppose that the total fixed costs add to \$0.75/lb. and the total variable costs add to \$2.25/lb. Let  $S_1$  be the price of copper per pound in one year.

- ▶ The net income per pound if the mine is open is  $S_1 - 0.75 - 2.25 = S_1 - 3.00$ .
- ▶ The net income per pound if the mine is closed is  $-0.75$ /lb.

Utah Red Mountain Company would be better to close its mine if  $S_1 - 3 \leq -0.75$ , which is equivalent to  $S_1 \leq 2.25$ . URM would be better keep its mine open if  $S_1 \geq 2.25$ . But, if  $2.25 \leq S_1 \leq 3$ , URM assumes the loss  $3 - S_1$ . If  $S_1 \geq 3$  and URM keeps its mine open, its (positive) net income is  $S_1 - 3$ . The following table shows the net income of URM (see also Figure 1).



**Figure 1:** Profit according whether the mine is closed, insured open and open insured with a forward.

Suppose that URMCo can enter a short forward contract agreeing to sell its copper one year from now. Suppose that  $F_{0,1} = 3.5/\text{lb}$ . If Utah Red Mountain Company enters this contract, its profit is  $\$0.5/\text{lb}$ . In this way Utah Red Mountain Company reduces risk. Figure 1 shows the graph of the profit under the three considered alternatives. For an uninsured position, the possible losses can be very high. However, by entering the forward, the company has a fixed benefit.

However, URMCo would like to benefit if the price of the copper goes higher than \$3.5/lb. It could use options. Suppose that the continuous rate of interest is 0.05 and the variability is  $\sigma = 0.25$ . Using the Black–Scholes formula, we have the following table of premiums of options:

Table 1:

$K$	3.3	3.4	3.5	3.6	3.7	3.8	3.9
Call( $K, T$ )	0.42567	0.3762	0.33119	0.29048	0.25386	0.22111	0.19196
Put( $K, T$ )	0.23542	0.28107	0.33119	0.3856	0.44411	0.50648	0.57245

Utah Red Mountain Company could buy a 3.7–strike put. URMC's profit per pound is

$$\begin{aligned} & S_1 - 3 + \max(3.7 - S_1, 0) - (0.44411)e^{0.05} \\ &= \max(3.7, S_1) - 3.466880007 \\ &= \begin{cases} 0.2331199934 & \text{if } S_1 < 3.7, \\ S_1 - 3.466880007 & \text{if } 3.7 \leq S_1. \end{cases} \end{aligned}$$

Under this strategy, URMC does not benefit much if the price of copper is high. Another strategy is to buy a 3.4–strike put. The profit per pound is

$$\begin{aligned}
 & S_1 - 3 + \max(3.4 - S_1, 0) - (0.28107)e^{0.05} \\
 &= \max(3.4, S_1) - 3.295480767 \\
 &= \begin{cases} 0.104519233 & \text{if } S_1 < 3.4, \\ S_1 - 3.295480767 & \text{if } 3.4 \leq S_1. \end{cases}
 \end{aligned}$$

Under this strategy, the company makes \$0.17139924/lb more than before if the price of copper is over \$3.7/oz. However, its guaranteed profit is 0.104519233, which is smaller than the guaranteed profit under a 3.7–strike put.

See Figure 2 for the profit under a short forward, a 3.7–strike put and 3.4–strike put.

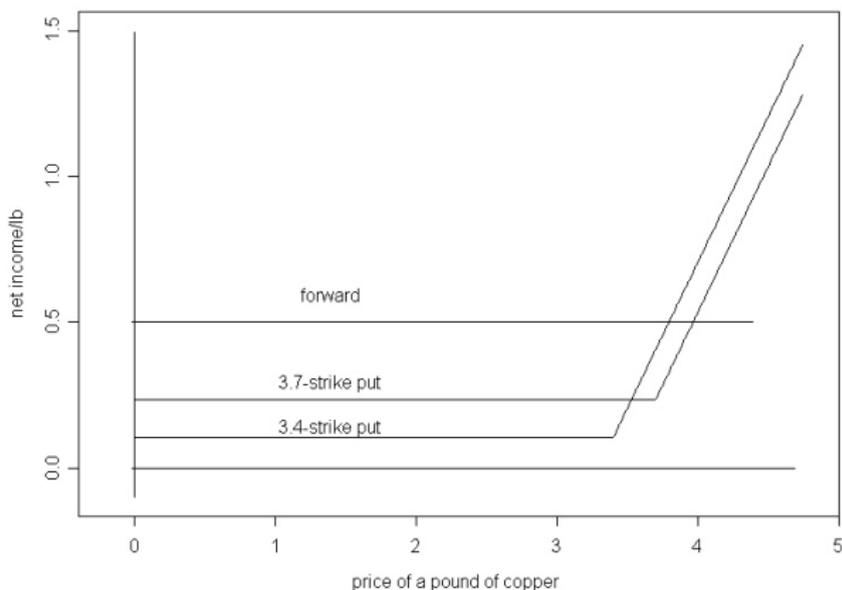


Figure 2: Profit for forward, 3.4–strike put and 3.7–strike put.

In other words, buying a put is like buying insurance against small spot prices. As bigger the strike price as bigger the price of the insurance. As bigger the strike price as bigger the obtained payment when the insurance is needed. A producer company needs to buy a put with a strike large enough strike to cover low spot prices. If the strike put is too large, the company will be wasting money in insurance which it does not need.

## Example 4

*Utah Red Mountain Company has the following profits:*

*(i) If it does not insure,  $S_1 - 3.00$ .*

*(ii) If it enters a short forward, 0.5.*

*(iii) If it buys a 3.4–strike put,  $\max(3.4, S_1) - 3.295480767$ .*

*Suppose that an actuary consulting for Utah Red Mountain Company estimates that the price of cooper in one year will be either 2.6, or 3.6, or 4.6, with respective probabilities 0.36, 0.33 and 0.31.*

*(i) Compute the URMC's expected profit before taxes for each of the above strategies. Find the strategy with the biggest expected profit before taxes.*

*(ii) The URMC pays a 40% tax rate, and has no tax benefits for losses. Compute the URMC's expected profit after taxes for each of the above strategies. Find the strategy with the biggest expected profit after taxes.*

**Solution:** (i)

uninsured profit	-0.4	0.6	1.6
profit under a short forward	0.5	0.5	0.5
profit under a 3.4-strike put	0.104519233	0.304519233	1.304519233
$S_T$	2.6	3.6	4.6
Probability	0.36	0.33	0.31

The expected profit under an uninsured position is

$$(-0.40)(0.36) + (0.60)(0.33) + (1.60)(0.31) = 0.55.$$

The expected profit under a short forward is 0.5. The expected profit under a 3.7-strike put is

$$(0.104519233)(0.36) + (0.304519233)(0.33) + (1.304519233)(0.31) = 0.542519233.$$

The strategy with the biggest expected profit before taxes is the uninsured position.

**Solution:** (ii) After taxes, the profits are given by the following table:

uninsured profit	-0.40	0.36	0.96
profit with a forward	0.3	0.3	0.3
profit with a 3.4-strike put	0.0627115398	0.1827115398	0.7827115398
Probability	0.36	0.33	0.31

The expected profit after taxes for an uninsured position is

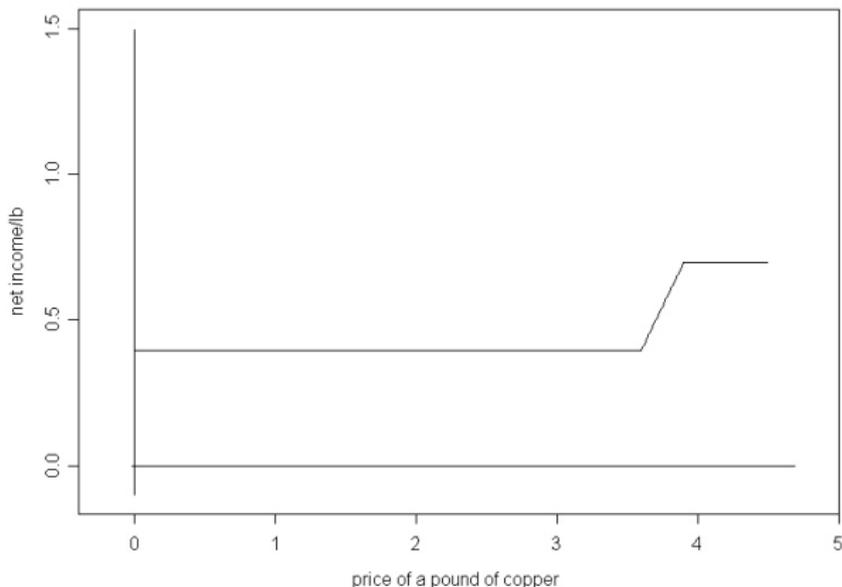
$$(-0.40)(0.36) + (0.36)(0.33) + (0.96)(0.31) = 0.2724.$$

The expected profit after taxes under a short forward is  $(0.6)(0.5) = 0.3$ . The expected profit after taxes under a a 3.4-strike put is  $(0.6)(0.542519233) = 0.3255115398$ .

Another strategy for URM is to buy a 3.6–strike put and sell a 3.9–strike call option. Buying these two options, URM will sell cooper at  $\min(\max(3.6, S_1), 3.9)$ . This strategy is called a collar. URM's profit per pound is

$$\begin{aligned}
 & S_1 - 3 + \max(3.6 - S_1, 0) - \max(S_1 - 3.9, 0) \\
 & - (0.3856 - 0.19196)e^{0.05} \\
 = & -3 + 3.9 + \max(3.6, S_1) - \max(S_1, 3.9) - (0.3856 - 0.19196)e^{0.05} \\
 = & -3 + 3.9 + \max(3.6, S_1) - \max(S_1, 3.9, 3.6) - (0.3856 - 0.19196)e^{0.05} \\
 = & \min(\max(3.6, S_1), 3.9) - 3 \\
 & - (0.3856 - 0.19196)e^{0.05} \\
 = & \min(\max(3.6, S_1), 3.9) - 3.203568135 \\
 = & \begin{cases} 0.396431865 & \text{if } S_1 < 3.6, \\ S_1 - 3.203568135 & \text{if } 3.6 \leq S_1 < 3.9, \\ 0.696431865 & \text{if } 3.9 < S_T. \end{cases}
 \end{aligned}$$

Under this strategy the profit of the company is very close to that of a forward contract. But, instead of winning a constant of \$0.5/lb. in the forward contract, the company's profit varies with the future price of copper, although not much.



Another type of strategies are the ones called **paylater** strategies. By a buying a put, URMH hedges against low prices. But, if the prices are high, its profit is not as high as when it does not hedge. A paylater strategy allows to have the usual profit for high enough spot prices. It is like the price of the insurance does not need to be paid. Consider the strategy of buying two 3.6–strike puts and buying a  $K$ –strike call, such that the cost of the portfolio is zero. The cost of two 3.6–strike puts is  $(2)(0.3856) = 0.7712$ . By numerical methods, we have that the cost of a 4.1763–strike put is 0.7712. Hence,  $K = 4.1763$ . The profit of this strategy is

$$\begin{aligned}
 & S_1 - 3 + 2 \max(3.6 - S_1, 0) - \max(4.1763 - S_1, 0) \\
 &= -3 + 2 \max(3.6, S_1) - \max(4.1763, S_1) \\
 &= \begin{cases} 0.0237 & \text{if } S_1 < 3.6, \\ 2S_1 - 7.1763 & \text{if } 3.6 \leq S_1 < 4.1763, \\ S_1 - 3 & \text{if } 4.1763 < S_1, \end{cases}
 \end{aligned}$$

Notice that under the previous strategy Company URMC always has a positive profit and if the spot price is bigger than 4.1763, its profit is the same as it would not hedge.

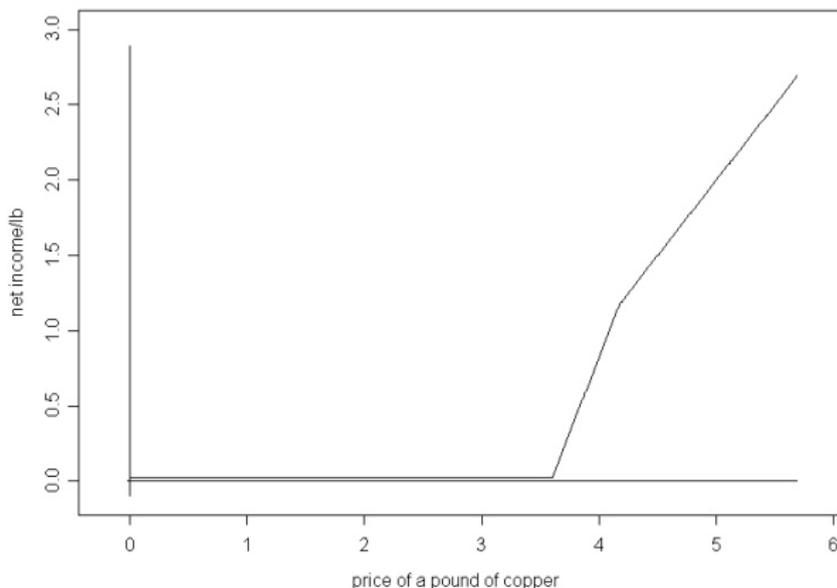


Figure 4: Profit for a paylater.

# The buyer's perspective on risk management.

Toughminum makes fridges. Suppose that:

- (i) The fixed cost per fridge is \$100.
- (ii) Toughminum sells fridges for \$350.
- (iii) To manufacture a fridge Toughminum need 5 pounds of aluminum.

Let  $S_T$  be the price of a pound of aluminum at time  $T$ . The profit one year from now is

$$350 - (5)(S_1) - 100 = 250 - 5S_1.$$

Figure 5 shows the graph of this profit ("uninsured" line).

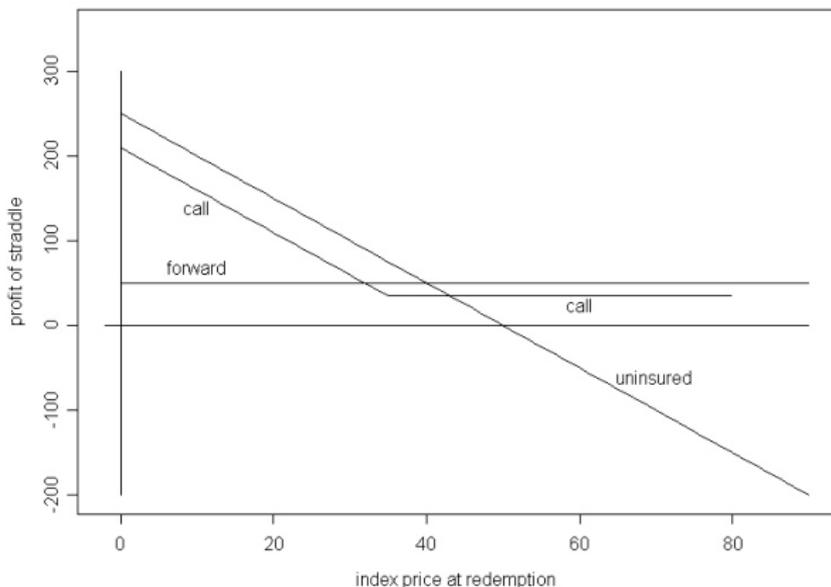


Figure 5: Profit uninsured, long forward and call.

Toughminum could face severe losses if the price of the aluminum goes very high. To hedge risk, it could enter a long forward. If aluminum is selling at \$40 a pound in the forward market, the profit of entering a long forward is

$$350 - (5)(40) - 100 = 50.$$

Figure 5 shows the graph of this profit ("forward" line).

Another alternative is to buy a call. Suppose that Toughminum buys a 35–strike call with an expiration date one year from now and nominal amount 5 lbs. Assume that  $\sigma = 0.35$  and  $r = 0.06$ . Then,  $\text{Call}(35, 1) = 7.609104$ . The profit is

$$\begin{aligned}
 & 250 - 5S_1 + 5 \max(S_1 - 35, 0) - 5\text{Call}(35, 1)e^r \\
 &= 250 + 5 \max(-35, -S_1) - 5(7.609104)e^{0.06} \\
 &= 209.6018764 - 5 \min(35, S_1) \\
 &= \begin{cases} 209.6018764 - 5S_1 & \text{if } S_1 < 35, \\ 34.6018764 & \text{if } 35 \leq S_1. \end{cases}
 \end{aligned}$$

Figure 5 shows the graph of this profit ("call" line).

## Example 5

*Toughminum has the following profits:*

*(i) If it does not insure,  $250 - 5S_1$ .*

*(ii) If it enters a short forward 50.*

*(iii) If it buys a 35–strike call,  $209.6018 - 5 \min(35, S_1)$ .*

*Suppose that an actuary consulting for Toughminum estimates that the price of aluminum in one year will be either 30, 35, or 40, or 55, with respective probabilities 0.25, 0.25, 0.25, and 0.25.*

*(i) Compute the Toughminum's expected profit before taxes for each of the above strategies. Find the strategy with the biggest expected profit before taxes.*

*(ii) The Toughminum pays a 35% tax rate, and has no tax benefits for losses. Compute the Toughminum's expected profit after taxes for each of the above strategies. Find the strategy with the biggest expected profit after taxes.*

**Solution:** (i)

uninsured profit	100	75	50	-25
profit under a long forward	50	50	50	50
profit under a 35-strike call	59.6019	34.6019	34.6019	34.6019
$S_T$	30	35	40	55
Probability	0.25	0.25	0.25	0.25

The expected profit under an uninsured position is

$$(100)(0.25) + (75)(0.25) + (50)(0.25) + (-25)(0.25) = 50.$$

The expected profit under a long forward is 50. The expected profit under a 35-strike call is

$$(59.6019)(0.25) + (34.6019)(0.25) + (34.6019)(0.25) + (34.6019)(0.25) =$$

The strategies with the biggest expected payoff are the uninsured position and the long forward position.

**Solution:** (ii) After taxes, the profits are given by the following table:

uninsured profit	65.00	48.75	32.50	-25.00
profit with a long forward	32.5	32.5	32.5	32.5
profit under a 35-strike call	38.7412	22.4912	22.4912	22.4912
$S_T$	30	35	40	55
Probability	0.25	0.25	0.25	0.25

The expected profit for an uninsured position is

$$(65)(0.25) + (48.75)(0.25) + (32.5)(0.25) + (-25)(0.25) = 30.3125.$$

The expected profit under a long forward is 32.5. The expected profit under a 35-strike call is

$$(38.7412)(0.25) + (22.4912)(0.25) + (22.4912)(0.25) + (22.4912)(0.25) =$$

The strategy with the biggest expected payoff is the long forward position.

Toughminum could sell a 30–strike put option and buy a 45–strike call option. Both with nominal amount 5 lbs. This position is called a 30–45 reverse collar. Under this position, Toughminum will buy aluminum at  $\min(\max(30, S_1), 45)$  per lb. The cost of a 30–strike put is \$1.308289/lb. The cost of a 45–strike call is \$3.514166/lb. The profit per ounce is

$$\begin{aligned}
 & 350 - 100 - 5S_1 - 5 \max(30 - S_1, 0) + 5 \max(S_1 - 45, 0) \\
 & + 5\text{Put}(30, 1)e^r - 5\text{Call}(45, 1)e^r \\
 = & 250 - 5 \max(30, S_1) - 5(45) + 5 \max(S_1, 45) \\
 & + 5(1.308289)e^{0.06} - 5(3.514166)e^{0.06} \\
 = & 238.2886 - 5 \max(30, S_1) - 5(45) + 5 \max(S_1, 45, 30) \\
 = & 238.2886 - 5 \min(\max(30, S_1), 45)
 \end{aligned}$$

or

$$\text{profit} = \begin{cases} 88.2886 & \text{if } S_1 < 30, \\ 238.2886 - 5S_1 & \text{if } 30 \leq S_1 < 45, \\ 13.2886 & \text{if } 45 \leq S_1. \end{cases}$$

The graph of this profit in Figure 6.

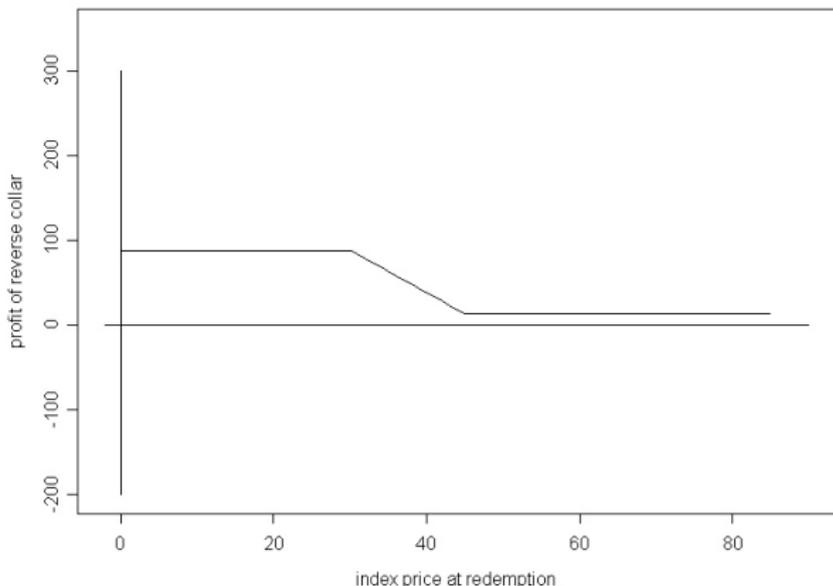


Figure 6: Profit for a 30–45 reverse collar.

## Importer/exporter's perspective.

Suppose that Company ABCD imports electronics to Canada. It gets paid in Canadian dollars. Whenever this company gets a payment, it needs to exchange Canadian dollars into US dollars. Suppose that in two months, ABCD expects to get 100,000 Canadian dollars. If the price of a Canadian dollar at time  $T$  is  $S_T$ , the amount of US dollars the company ABCD will get is  $\frac{100000}{S_{2/12}}$ . To hedge against possible changes in exchange rates, company ABCD can sell a forward on 100000 Canadian dollars at the current price. It also can buy a put on Canadian dollars.

## Example 6

*Suppose that Company ABCD buys a put on (Canadian \$'s) CAD100000 with a strike price of (U.S.A. dollar) USD0.85 per CAD for USD0.01 per CAD. Two months later, ABCD receives CAD100000. At this moment the exchange rate is USD0.845 per CAD.*

*(i) How many US dollars does company ABCD gets in this transaction?*

*(ii) How many US dollars would company ABCD have gotten in the exchange if it would not have signed the put?*

## Example 6

Suppose that Company ABCD buys a put on (Canadian \$'s) CAD100000 with a strike price of (U.S.A. dollar) USD0.85 per CAD for USD0.01 per CAD. Two months later, ABCD receives CAD100000. At this moment the exchange rate is USD0.845 per CAD.

(i) How many US dollars does company ABCD gets in this transaction?

(ii) How many US dollars would company ABCD have gotten in the exchange if it would not have signed the put?

**Solution:** (i) Since the strike price is bigger than the current spot price, the company exercises the put. It gets  $(100000)(0.85 - 0.01) = 84000$  US dollars.

(i) Company ABCD would have got  $100000(0.845) = 84500$  US dollars