Manual for SOA Exam MLC.

Chapter 5. Life annuities.

Review.

Extract from:
"Arcones' Manual for the SOA Exam MLC. Spring 2010 Edition".
available at http://www.actexmadriver.com/

Whole life annuity due

Payments at $0, 1, \dots, K_x - 1$,

$$\ddot{Y}_x = \frac{1 - v^{K_x}}{d} = \frac{1 - Z_x}{d},$$
$$\ddot{a}_x = \sum_{k=1}^{\infty} \frac{1 - v^k}{d} \cdot {}_{k-1|}q_x = \frac{1 - A_x}{d} = \ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x.$$

and

$$\mathrm{Var}(\ddot{Y}_x) = \frac{\mathrm{Var}(Z_x)}{d^2} = \frac{{}^2A_x - A_x^2}{d^2},$$

Under constant force of mortality, $\ddot{a}_x = \frac{1}{1 - vp_x}$.

Whole life annuity immediate

Payments at $1, \dots, K_x - 1$,

$$Y_x = \ddot{Y}_x - 1.$$

Whole life continuous annuity

Continuous payments from 0 to T_x

$$\overline{Y}_x = \overline{a}_{\overline{T_x|}} = \frac{1 - v^{T_x}}{\delta} = \frac{1 - \overline{Z}_x}{\delta},$$

$$\overline{a}_x = E\left[\frac{1 - \overline{Z}_x}{\delta}\right] = \frac{1 - \overline{A}_x}{\delta} = \int_0^\infty v^t \cdot {}_tp_x \, dt,$$

$$\mathrm{Var}(\overline{Y}_x) = \mathrm{Var}\left(\frac{1 - \overline{Z}_x}{\delta}\right) = \frac{\mathrm{Var}(\overline{Z}_x)}{\delta^2} = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2}.$$

Under constant force of mortality μ, $\overline{a}_x = \frac{1}{\mu+\delta}$.

Due n–year deferred annuity

If $K_x \geq n+1$, unit payments at times $n, \cdots, K_x - 1$ are made.

$$
\begin{aligned}
{}_{n|}\ddot{Y}_x &= v^n \ddot{a}_{\overline{K_x - n|}} I(K_x > n), \\
{}_{n|}\ddot{a}_x &= \sum_{k=n+1}^{\infty} v^n \ddot{a}_{\overline{k-n|}} \cdot {}_{k-1|}q_x = \sum_{k=n}^{\infty} v^k \cdot {}_k p_x = {}_nE_x \ddot{a}_{x+n}.
\end{aligned}
$$

Under constant force of mortality μ, ${}_{n|}\ddot{a}_x = \frac{v^n p_x^n}{1 - v p_x} = \frac{e^{-n(\delta+\mu)}}{1 - e^{-(\delta+\mu)}}$.

Due n–year deferred immediate

If $K_x \geq n+2$, unit payments at times $n+1, \cdots, K_x - 1$ are made.

$$
{}_{n|}Y_x = \sum_{k=n+1}^{\infty} Z_{x:\overline{k}|}^{\;\,1} = {}_{n+1|}\ddot{Y}_x,
$$

$$
{}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k \cdot {}_k p_x = {}_{n+1|}\ddot{a}_x = {}_nE_x a_{x+n}.
$$

n–year deferred annuity continuous

If $T_x > n$, continuous payments in the interval $[n, T_x]$.

$$
{}_{n|}\overline{Y}_x = \int_n^{T_x} v^s \, ds I(T_x > n) = v^n \overline{a}_{\overline{T_x - n}|} I(T_x > n)
$$

$$
= \int_n^{\infty} v^n \overline{a}_{\overline{t-n}|} \cdot {}_t p_x \cdot \mu_{x+t} \, dt = {}_n E_x \cdot \overline{a}_{x+n}
$$

Under constant force of mortality,

$$
{}_{n|}\overline{a}_x = \frac{e^{-n(\mu+\delta)}}{\mu + \delta}.
$$

Due n–year temporary annuity.

If $K_x \le n$, benefit payments are made at times $0, 1, \ldots, K_x - 1$.

$$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|},$$

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=1}^{n} \ddot{a}_{\overline{k}|} \cdot {}_{k-1|}q_x + \ddot{a}_{\overline{n}|} \cdot {}_{n}p_x$$

$$= \sum_{k=1}^{n-1} \ddot{a}_{\overline{k}|} \cdot {}_{k-1|}q_x + \ddot{a}_{\overline{n}|} \cdot {}_{n-1}p_x = \frac{1 - A_{x:\overline{n}|}}{d} = \sum_{k=0}^{n-1} v^k {}_{k}p_x,$$

$$\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n}E_x \ddot{a}_{x+n},$$

$$\mathrm{Var}(\ddot{Y}_{x:\overline{n}|}) = \frac{{}^{2}A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2}.$$

Under constant force of mortality,

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - v^n p_x^n}{1 - v p_x}.$$

The actuarial accumulated value at time n of n–year term temporary due annuity is defined by

$$\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} = \frac{\ddot{a}_{x:\overline{n}|}}{v^n\,{}_np_x} = \frac{\sum_{k=0}^{n-1} v^k\,{}_kp_x}{v^n\,{}_np_x}$$

$$= \sum_{k=0}^{n-1} \frac{1}{v^{n-k}\,{}_{n-k}p_{x+k}} = \sum_{k=0}^{n-1} \frac{1}{{}_{n-k}E_{x+k}}.$$

Immediate n–year temporary annuity

If $2 \le K_x \le n+1$, payments are made at times $1, \dots, K_x - 1$.

$$Y_{x:\overline{n}|} = a_{\overline{\min(K_x-1,n)}|} = \ddot{Y}_{x:\overline{n+1}|} - 1,$$
$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n+1}|} - 1 = a_x = {}_{n}|a_x + a_{x:\overline{n}|} = {}_{n}|a_x + {}_{n}E_x a_{x+n},$$
$$\mathrm{Var}(Y_{x:\overline{n}|}) = \mathrm{Var}(\ddot{Y}_{x:\overline{n+1}|})$$

Continuous n–year temporary annuity

Continuous payments in the interval $[0, \min(T_x, n)]$.

$$\overline{Y}_{x:\overline{n}|} = \overline{a}_{\overline{\min(T_x,n)}|} = \frac{1 - \overline{Z}_{x:\overline{n}|}}{\delta},$$

$$\overline{a}_{x:\overline{n}|} = \int_0^n \overline{a}_{\overline{t}|} \cdot {}_tp_x \mu_{x+t}\, dt + \overline{a}_{\overline{n}|} \mathbb{P}\{T_x > n\} = \int_0^n v^t \cdot {}_tp_x\, dt,$$

$$\mathrm{Var}(\overline{Y}_{x:\overline{n}|}) = \frac{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}{\delta^2}.$$

Under constant force of mortality,

$$\overline{a}_{x:\overline{n}|} = \frac{1 - e^{-n(\mu+\delta)}}{\mu + \delta}.$$

$$\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_{n}|\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_{n}E_x \overline{a}_{x+n},$$

$$\overline{a}_{x:\overline{m+n}|} = \overline{a}_{x:\overline{n}|} + {}_{m}E_x \cdot \overline{a}_{x+m:\overline{n}|}.$$

The actuarial accumulated value at time n of an n–year term temporary continuous annuity is

$$\bar{s}_{x:\overline{n}|} = \frac{\bar{a}_{x:\overline{n}|}}{{}_nE_x} = \frac{\bar{a}_{x:\overline{n}|}}{v^n \cdot {}_np_x} = \frac{\int_0^n v^t \cdot {}_tp_x \, dt}{v^n \cdot {}_np_x}$$
$$= \int_0^n \frac{1}{v^{n-t} \cdot {}_{n-t}p_{x+t}} \, dt = \int_0^n \frac{1}{{}_{n-t}E_{x+t}} \, dt.$$

$\frac{1}{{}_{n-t}E_{x+t}}$ is the actuarial factor from time t to time n for a live age x.

n–year certain life annuity–due

Payments are made at times $0, 1, \ldots, \max(n-1, K_{x-1})$,

$$\ddot{Y}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{\max(n,K_x)}|} = \ddot{a}_{\overline{n}|} + {}_{n}|\ddot{Y}_x,$$

$$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_{n}|\ddot{a}_x,$$

$$\mathrm{Var}(\ddot{Y}_{\overline{x:\overline{n}|}}) = \mathrm{Var}({}_{n}|\ddot{Y}_x).$$

n–year certain life annuity–immediate

Payments are made at times $1, \dots, \max(n, K_x - 1)$,

$$Y_{\overline{x:\overline{n}|}} = a_{\overline{\max(n,K_{x-1})}|} = a_{\overline{n}|} + {}_n|Y_x = \ddot{Y}_{\overline{x:\overline{n+1}|}} - 1,$$

$$a_{\overline{x:\overline{n}|}} = a_{\overline{n}|} + {}_n|a_x = \ddot{a}_{\overline{x:\overline{n+1}|}} - 1,$$

$$\mathrm{Var}(Y_{\overline{x:\overline{n}|}}) = \mathrm{Var}({}_n|Y_x).$$

n–year certain life continuous annuity

Continuous payments at unit rate are made in the interval $[0, \max(n, T_x)]$.

$$\overline{Y}_{\overline{x:\overline{n}|}} = \overline{a}_{\overline{\max(n,T_x)}|} = \overline{a}_{\overline{n}|} + {}_{n}|\overline{Y}_x,$$

$$\overline{a}_{\overline{x:\overline{n}|}} = \overline{a}_{\overline{n}|} + {}_{n}|\overline{a}_x = \overline{a}_{\overline{n}|} + \int_n^\infty v^t \cdot {}_tp_x \, dt,$$

$$\mathrm{Var}(\overline{Y}_{\overline{x:\overline{n}|}}) = \mathrm{Var}({}_{n}|\overline{Y}_x).$$

Whole life annuity due paid m times a year

Let $J_x^{(m)} = \lceil m T_x \rceil$, i.e. $J_x^{(m)}$ is the integer such that if $T_x \in \left(\frac{J_x^{(m)}-1}{m}, \frac{J_x^{(m)}}{m} \right]$.
A whole life annuity due paid m times a year makes payments of $\frac{1}{m}$ at times $0, \frac{1}{m} \dots \frac{J_x^{(m)}-1}{m}$.

$$\ddot{Y}_x^{(m)} = \frac{1}{m} \ddot{a}_{\overline{J_x^{(m)}}|\frac{i^{(m)}}{m}} = \frac{1 - Z_x^{(m)}}{d^{(m)}} = \frac{1}{m} \sum_{k=0}^{\infty} Z_{x:\overline{\frac{k}{m}}|}^{\;\;1},$$

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1}{m} \sum_{k=0}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(\ddot{Y}_x^{(m)}) = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}.$$

Whole life annuity immediate paid m times a year.

Payments of $\frac{1}{m}$ at times $\frac{1}{m}, \frac{2}{m} \dots \frac{J_x^{(m)}-1}{m}$.

$$Y_x^{(m)} = \ddot{Y}_x^{(m)} - \frac{1}{m} = \frac{1}{m} a_{\overline{J_x^{(m)}-1}|\frac{i^{(m)}}{m}} = \frac{1}{m} \sum_{k=1}^{\infty} Z_{x:\frac{k}{m}|}^{\;\;\;\frac{1}{}},$$

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} = \frac{1}{m} \sum_{k=1}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(Y_x^{(m)}) = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}.$$

n–year term life annuity due paid m times a year..

Payments of $\frac{1}{m}$ at times $0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{\min(J_x^{(m)}, nm)-1}{m}$.

$$\ddot{Y}^{(m)}_{x:\overline{n}|} = \frac{1}{m}\ddot{a}_{\overline{\min(J_x^{(m)}, nm)}|\frac{i^{(m)}}{m}} = \frac{1 - Z^{(m)}_{x:\overline{n}|}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{nm-1} Z_{x:\overline{\frac{k}{m}}|},$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{nm-1} v^{\frac{1}{m}} \cdot {}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(\ddot{Y}^{(m)}_{x:\overline{n}|}) = \frac{\mathrm{Var}(Z^{(m)}_{x:\overline{n}|})}{(d^{(m)})^2}.$$

n–year term annuity immediate paid m times a year.

Payments of $\frac{1}{m}$ at times $\frac{1}{m}, \frac{2}{m}, \dots, \frac{\min(J_x^{(m)-1}, nm)}{m}$.

$$Y_{x:\overline{n}|}^{(m)} = \frac{1}{m} a_{\overline{\min(J_x^{(m)}-1, nm)}|\frac{i^{(m)}}{m}} = \frac{1}{m} \sum_{k=1}^{nm} Z_{x:\frac{1}{\frac{k}{m}}|} = \ddot{Y}_{x:\overline{n}|}^{(m)} - \frac{1}{m} + \frac{1}{m} Z_{x:\frac{1}{\overline{n}|}},$$

$$a_{x:\overline{n}|}^{(m)} = \frac{1}{m} \sum_{k=1}^{nm} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m} + \frac{1}{m} \cdot {}_nE_x.$$

Due n–year deferred annuity paid m times a year..

If $J_x^{(m)} > nm$, payments of $\frac{1}{m}$ are made at times $\frac{nm}{m}, \frac{nm+1}{m}, \dots, \frac{J_x^{(m)}-1}{m}$.

$$
\begin{aligned}
{}_{n|}\ddot{Y}_x^{(m)} &= \frac{1}{m} \sum_{k=nm}^{\infty} Z_{x:\frac{1}{\frac{k}{m}}|}, \\
{}_{n|}\ddot{a}_x^{(m)} &= {}_nE_x \cdot \ddot{a}_{x+n}^{(m)}, \\
\ddot{a}_x^{(m)} &= \ddot{a}_{x:\overline{n}|}^{(m)} + {}_{n|}\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + {}_nE_x \ddot{a}_{x+n}^{(m)}.
\end{aligned}
$$

Immediate n–year deferred annuity paid m times a year.

If $J_x^{(m)} > nm + 1$, payments of $\frac{1}{m}$ are made at times $\frac{nm+1}{m}, \frac{nm+2}{m}, \dots, \frac{J_x^{(m)}-1}{m}$.

$$
{}_{n|}Y_x^{(m)} = \frac{1}{m} \sum_{k=nm+1}^{\infty} Z_{x:\frac{1}{\frac{k}{m}|}}
$$

$$
= {}_{n|}\ddot{Y}_x^{(m)} - \frac{1}{m} Z_{x:\overline{n}|}^{\ \ 1},
$$

$$
{}_{n|}a_x^{(m)} = \frac{1}{m} \sum_{k=nm+1}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x = {}_nE_x \cdot a_{x+n}^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m} {}_nE_x,
$$

$$
a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_{n|}a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_nE_x a_{x+n}^{(m)}.
$$

Increasing life due annuities.

A unit annually increasing due whole life annuity has payments $1, 2, \ldots,$ at the beginning of the year and actuarial present value

$$(I\ddot{a})_x = \sum_{k=0}^{\infty} (k+1)v^k \cdot {}_kp_x.$$

A unit annually increasing n–year term due life unit annuity has payments $1, 2, \ldots, n$ at the beginning of the year and actuarial present value

$$(I\ddot{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1} (k+1)v^k \cdot {}_kp_x.$$

Increasing life immediate annuities.

A unit annually increasing immediate whole life annuity has payments $1, 2, \dots,$ at the end of the year and actuarial present value

$$(Ia)_x = \sum_{k=1}^{\infty} k v^k \cdot {}_k p_x.$$

A unit annually increasing n–year term life immediate annuity has payments $1, 2, \dots, n$ at the end of the year and actuarial present value

$$(Ia)_{x:\overline{n}|} = \sum_{k=1}^{n} k v^k \cdot {}_k p_x.$$

Decreasing life annuities.

A unit annually decreasing n–year term due life annuity due has payments $n, n-1, \dots, 1$ at the beginning of the year and actuarial present value

$$(D\ddot{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1} (n-k) v^k \cdot {}_k p_x.$$

A unit annually decreasing n–year term life immediate annuity has payments $n, n-1, \dots, 1$ at the end of the year and actuarial present value

$$(Da)_{x:\overline{n}|} = \sum_{k=1}^{n} (n+1-k) v^k \cdot {}_k p_x.$$

Continuously increasing life annuities.

A continuously increasing whole life unit annuity paid at the time of death has an actuarial present value of

$$(\overline{I}\overline{a})_x = \int_0^{\infty} t \cdot v^t \cdot {}_tp_x \, dt.$$

A continuously increasing n–year term life unit annuity paid at the time of death with rate of payments t has an actuarial present value of

$$(\overline{I}\overline{a})_{x:\overline{n}|} = \int_0^{n} t \cdot v^t \cdot {}_tp_x \, dt.$$

Annually increasing life annuities.

An annually increasing whole life unit annuity paid at the time of death has an actuarial present value of

$$(I\bar{a})_x = \int_0^\infty \lceil t \rceil \cdot v^t \cdot {}_tp_x \, dt.$$

An annually increasing n–year term life unit annuity paid at the time of death has an actuarial present value of

$$(I\bar{a})_{x:\overline{n}|} = \int_0^n \lceil t \rceil \cdot v^t \cdot {}_tp_x \, dt.$$

Continuous decreasing life annuities.

A continuously increasing decreasing n–year term life unit annuity paid at the time of death has an actuarial present value of

$$(\overline{D}\overline{a})_{x:\overline{n}|} = \int_0^n (n-t) \cdot v^t \cdot {}_tp_x \, dt.$$

An annually decreasing n–year term life unit annuity paid at the time of death has an actuarial present value of

$$(D\overline{a})_{x:\overline{n}|} = \int_0^n \lceil n-t \rceil \cdot v^t \cdot {}_tp_x \, dt.$$

To find the different values of $a^{(m)}$ and $\overline{a}$, we interpolate the corresponding values of $A^{(m)}$ and $\overline{A}$ and get the values of $a^{(m)}$ and $\overline{a}$, using relations between a's and A's.

Theorem 1

Assuming a uniform distribution of deaths, we have that:
(i) $\overline{A}_x = \frac{i}{\delta} A_x$.
(ii) $\overline{A}^{\,1}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|}$.
(iii) ${}_{n}|\overline{A}_x = \frac{i}{\delta} \cdot {}_{n}|A_x$.
(iv) $\overline{A}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$.

Theorem 2

Assuming a uniform distribution of deaths, we have that:
(i) $A^{(m)}_x = \frac{i}{i^{(m)}} A_x$.
(ii) $\overset{(m)}{A}{}^{1}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}$.
(iii) ${}_{n}|A^{(m)}_x = \frac{i}{i^{(m)}} \cdot {}_{n}|A_x$.
(iv) $A^{(m)}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$.

