1 Interest Theory

$A(t)$ is the **amount function**. $a(t)$ is the **accumulation function**. $a(t) = \frac{A(t)}{A(0)}$.

$$k \text{ at time } s \equiv \frac{kA(t)}{A(s)} \text{ at time } t.$$

$v_t = \frac{1}{a(t)}$, $t \geq 0$, is called the discount function **discount function**.

$$k \text{ at time } s \equiv \frac{kv_t}{v_s} \text{ at time } t.$$

Under compound interest $v_t = (1+i)^{-t}$. i is the **annual effective rate of interest**. $1+i$ is the **one year interest factor**. $\nu = 1 - d$ is the **one year discount factor**. d is the **annual rate of discount**.

$$i = \frac{d}{1-d} = \frac{1-\nu}{\nu},\ d = \frac{i}{i+1} = 1-\nu,\ \frac{1}{1+i} = 1-d = \nu,\ i\nu = d, 1 = (1-d)(1+i).$$

$i^{(m)}$ is the **nominal rate of interest compounded** m **times a year**. $d^{(m)}$ is the **nominal rate of discount compounded** m **times a year**.

$$1+i = \left(1 + \frac{i^{(m)}}{m}\right)^m = (1-d)^{-1} = \left(1 - \frac{d^{(m)}}{m}\right)^{-m}.$$

The **force of interest** is

$$\delta_t = -\frac{d}{dt}\ln(v_t) = \frac{d}{dt}\ln a(t) = \frac{a'(t)}{a(t)} = \frac{d}{dt}\ln A(t) = \frac{A'(t)}{A(t)}.$$

$$v_t = e^{-\int_0^t \delta_s\, ds},\ a(t) = e^{\int_0^t \delta_s\, ds}.$$

Annuities

The cashflow, present and future values of an **annuity–due** with level payments of one are:

Contributions	1	1	1	$\cdots$	1	0
Time	0	1	2	$\cdots$	$n-1$	n

$$\ddot{a}_{\overline{n}|i} = \frac{1-\nu^n}{d} \text{ and } \ddot{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d}.$$

The cashflow, present and future values of an **annuity–immediate** with level payments of one:

Contributions	0	1	1	$\cdots$	1
Time	0	1	2	$\cdots$	n

$$a_{\overline{n}|i} = \frac{1-\nu^n}{i} \text{ and } s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}.$$

The cashflow and present value of an **perpetuity–due** with level payments of one are:

Contributions	1	1	1	$\cdots$
Time	0	1	2	$\cdots$

$$\ddot{a}_{\overline{\infty}|i} = \frac{1}{d}.$$

The cashflow and present value of a **perpetuity–immediate** with level payments of one are:

Contributions	0	1	1	$\cdots$
Time	0	1	2	$\cdots$

$$a_{\overline{\infty}|i} = \frac{1}{i}.$$

The cashflow and present value of a **geometric annuity–due** with first payment of one are:

Payments	1	$1+r$	$(1+r)^2$	$\cdots$	$(1+r)^{n-1}$
Time	0	1	2	$\cdots$	$n-1$

and

$$(G\ddot{a})_{\overline{n}|i,r} = \ddot{a}_{\overline{n}|\frac{i-r}{1+r}}.$$

The cashflow and present value of a **geometric annuity–immediate** with first payment of one are:

Payments	1	$1+r$	$(1+r)^2$	$\cdots$	$(1+r)^{n-1}$
Time	1	2	3	$\cdots$	n

and

$$(Ga)_{\overline{n}|i,r} = \frac{1}{1+r} a_{\overline{n}|\frac{i-r}{1+r}}.$$

The cashflow and present value of a **geometric perpetuity–due** with first payment of one are:

Payments	1	$1+r$	$(1+r)^2$	$\cdots$	$(1+r)^{n-1}$	$\cdots$
Time	0	1	2	$\cdots$	n	$\cdots$

and

$$(G\ddot{a})_{\overline{\infty}|i,r} = \begin{cases} \frac{1+i}{i-r} & \text{if } i > r, \\ \infty & \text{if } i \le r. \end{cases}$$

The cashflow and present value of a **geometric perpetuity–immediate** with first payment of one are:

Payments	1	$1+r$	$(1+r)^2$	$\cdots$	$(1+r)^{n-1}$	$\cdots$
Time	1	2	3	$\cdots$	n	$\cdots$

and

$$(Ga)_{\overline{\infty}|i,r} = \begin{cases} \frac{1}{i-r} & \text{if } i > r \\ \infty & \text{if } i \leq r. \end{cases}$$

The cashflow, present and future values of a **due increasing annuity** with first payment of one are:

Payments	1	2	3	$\cdots$	n
Time	0	1	2	$\cdots$	$n-1$

$$(I\ddot{a})_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{d} \text{ and } (I\ddot{s})_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{d}.$$

The cashflow, present and future values of an **immediate increasing annuity** with first payment of one are:

Payments	1	2	3	$\cdots$	n
Time	1	2	3	$\cdots$	n

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i} \text{ and } (Is)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i}.$$

The cashflow and present value of an **increasing due perpetuity** with first payment of one are:

Payments	1	2	3	$\cdots$
Time	0	1	2	$\cdots$

and $(I\ddot{a})_{\overline{\infty}|i} = \frac{1}{d^2}$.

The cashflow and present value of an **increasing immediate perpetuity** with first payment of one are:

Payments	1	2	3	$\cdots$
Time	1	2	3	$\cdots$

and $(Ia)_{\overline{\infty}|i} = \frac{1}{id}$.

The cashflow, present and future values of a **decreasing due annuity** with first payment of one are:

Payments	n	$n-1$	$n-2$	$\cdots$	1
Time	0	1	2	$\cdots$	$n-1$

$$(D\ddot{a})_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{d} \text{ and } (D\ddot{s})_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{d}.$$

The cashflow, present and future values of a **decreasing immediate annuity** with first payment of one are:

Payments	n	$n-1$	$n-2$	$\cdots$	1
Time	1	2	3	$\cdots$	n

$$(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i} \text{ and } (Ds)_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i}.$$

The cashflow, present and future values of a **due annuity paid m times a year** are

Contributions	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\cdots$	$\frac{1}{m}$	0
Time (in years)	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{m}{m}$	$\frac{m+1}{m}$	$\cdots$	$\cdots$	$\frac{nm-1}{m}$	$\frac{nm}{m}$

$$\ddot{a}^{(m)}_{\overline{n}|i} = \frac{1-\nu^n}{d^{(m)}} \text{ and } \ddot{s}^{(m)}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d^{(m)}}.$$

The cashflow, present and future values of an **immediate annuity paid m times a year** are

Contributions	0	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\cdots$	$\frac{1}{m}$
Time (in years)	0	$\frac{1}{m}$	$\frac{2}{m}$	$\cdots$	$\frac{m}{m}$	$\frac{m+1}{m}$	$\cdots$	$\cdots$	$\frac{nm}{m}$

$$a^{(m)}_{\overline{n}|i} = \frac{1-\nu^n}{i^{(m)}} \text{ and } s^{(m)}_{\overline{n}|i} = \frac{(1+i)^n - 1}{i^{(m)}}.$$

The present value of a **continuous annuity with rate $C(t)$** is

$$\int_0^t C(s)v^s \, ds.$$

The present value of a **continuous annuity with constant unit rate** is

$$\overline{a}_{\overline{n}|i} = \int_0^n v^t \, dt = \frac{1-v^n}{\delta}.$$

The present value of an **annually increasing continuous annuity** is

$$(I\bar{a})_{\overline{n}|i} = \int_0^n [t+1]v^t \, dt = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{\delta}.$$

The present value of a **continuously increasing annuity** is

$$\left(\bar{I}\bar{a}\right)_{\overline{n}|i} = \int_0^n tv^t \, dt = \frac{\bar{a}_{\overline{n}|i} - nv^n}{\delta}.$$

The present value of an **annually decreasing continuous annuity** is

$$(D\overline{a})_{\overline{n}|i} = \int_0^n [n+1-t]v^t \, dt = \frac{n - a_{\overline{n}|i}}{\delta}.$$

The present value of a **continuously decreasing continuous annuity** with is

$$\left(\overline{D}\overline{a}\right)_{\overline{n}|i} = \int_0^n (n-t)v^t \, dt = \frac{n - \overline{a}_{\overline{n}|i}}{\delta}.$$

2 Survival models.

The **cumulative distribution function** of the r.v. X is $F_X(x) = P\{X \le x\}$, $x \in \mathbb{R}$.

The **survival function** of the nonnegative r.v. X is $S_x(x) = s(x) = \Pr\{X > x\}$, $x \ge 0$.

If $h \ge 0$ and $H(x) = \int_0^x h(t)\,dt$, $x \ge 0$, then $E[H(X)] = \int_0^\infty s(t)h(t)\,dt$. In particular,

$$E[X] = \int_0^\infty s(t)\,dt,\ E[X^p] = \int_0^\infty s(t)pt^{p-1}\,dt,\ E[\min(X,a)] = \int_0^a s(t)\,dt.$$

If X is a discrete r.v.

$$E[H(X)] = \sum_{k=1}^\infty \Pr\{X \ge k\}(H(k) - H(k-1)).$$

In particular, for a positive integer a,

$$E[X] = \sum_{k=1}^\infty \Pr\{X \ge k\}, E[X^2] = \sum_{k=1}^\infty \Pr\{X \ge k\}(2k-1), E[\min(X,a)] = \sum_{k=1}^a \Pr\{X \ge k\}.$$

(x) is called a life–age–x. $T(x) = T_x = X - x$ is the **future lifetime of** (x).

The survival function of $T(x)$ is ${}_tp_x = \frac{s(x+t)}{s(x)}$, $t \ge 0$. The c.d.f. of $T(x)$ is ${}_tq_x = \frac{s(x)-s(x+t)}{s(x)}$, $t \ge 0$. We have that

$${}_tq_x = 1 - {}_tp_x,\ p_x = {}_1p_x,\ q_x = {}_1q_x,\ {}_{s|}{}_tq_x = \Pr\{s < T(x) \le s+t\} = {}_sp_x - {}_{s+t}p_x = {}_sp_x \cdot {}_tq_{x+s},$$

$${}_{m+n}p_x = {}_mp_x \cdot {}_np_{x+m},\ {}_np_x = p_xp_{x+1}\cdots p_{x+n-1},$$

$${}_{\sum_{j=1}^k n_j}p_x = {}_{n_1}p_x \cdot {}_{n_2}p_{x+n_1} \cdot {}_{n_3}p_{x+n_1+n_2} \cdots {}_{n_k}p_{x+\sum_{j=1}^{k-1} n_j}.$$

The **force of mortality** is $\mu(x) = \mu_x = -\frac{d}{dx}\ln S_X(x) = \frac{f_X(x)}{S_X(x)}$.

$$S_X(x) = \exp\left(-\int_0^x \mu(t)\,dt\right),\ {}_tp_x = e^{-\int_x^{x+t}\mu(y)dy}, f_{T(x)}(t) = {}_tp_x\mu(x+t).$$

$$\mathring{e}_0 = E[X] = \int_0^\infty {}_tp_0 dt,\ \mathring{e}_x = E[T(x)] = \int_0^\infty {}_tp_x dt,$$

$$\mathring{e}_{x:\overline{n}|} = E[\min(T(x),n)] = \int_0^n {}_tp_x\,dt,\ \mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$$

$\lceil t \rceil$ is the least integer greater than or equal to t, $\lceil t \rceil = k$ if $k-1 < t \le k$. K_x is the **time interval of death** of a life age x. $K(x)$ is the **curtate duration of death** of a life aged x, i.e. the **number of complete years lived** by this life.

$$K_x = \lceil T(x) \rceil, K(x) = K_x - 1,\ K(x) = \lceil T(x) \rceil - 1,$$

$$e_x = E[K(x)] = \sum_{k=1}^\infty {}_kp_x, E[(K(x))^2] = \sum_{k=1}^\infty (2k-1)\cdot {}_kp_x,$$

$$e_x = p_x(1+e_{x+1}), e_{x:\overline{n}|} = \sum_{k=1}^n {}_kp_x, e_x = e_{x:\overline{n}|} + {}_np_xe_{x+n},\ \mathring{e}_{x:\overline{m+n}|} = \mathring{e}_{x:\overline{m}|} + {}_mp_x\mathring{e}_{x+m:\overline{n}|}.$$

For de Moivre's law:

$$f_X(x) = \frac{1}{\omega}, S_X(x) = \frac{\omega - x}{\omega}, \mu(x) = \frac{1}{\omega - x}, \text{ for } 0 \le x < \omega,$$
$${}_tp_x = \frac{\omega - x - t}{\omega - x}, \ {}_tq_x = \frac{t}{\omega - x}, 0 \le t \le \omega - x,$$
$$\overset{\circ}{e}_x = \frac{\omega - x}{2}, \ \mathrm{Var}(T(x)) = \frac{(\omega - x)^2}{12}, \ e_x = \frac{\omega - x - 1}{2}, \ \mathrm{Var}(K(x)) = \frac{(\omega - x)^2 - 1}{12}$$

Under constant force of mortality μ:

$$S_X(x) = e^{-\mu x}, \ F_X(x) = 1 - e^{-\mu x}, \ f_X(x) = \mu e^{-\mu x}, \ \mu(x) = \mu, \ \text{for } x > 0,$$
$${}_tp_x = \Pr\{T(x) > t\} = \frac{s(x+t)}{s(x)} = e^{-\mu t},$$
$$\overset{\circ}{e}_x = \frac{1}{\mu}, \overset{\circ}{e}_{x:\overline{n}|} = \frac{1 - e^{-\mu n}}{\mu}, \mathrm{Var}(T(x)) = \frac{1}{\mu^2}, e_x = \frac{p_x}{q_x}, e_{x:\overline{n}|} = \frac{p_x(1 - p_x^n)}{q_x}, \mathrm{Var}(K(x)) = \frac{p_x}{q_x^2}.$$

3 Life tables.

ℓ_x denote the **number of individuals alive** at age x. The number of individuals which died between ages x and $x + t$ is ${}_td_x = \ell_x - \ell_{x+t}$. The number of individuals which died between ages x and $x + 1$ is $d_x = \ell_x - \ell_{x+1}$. We have that

$$s(x) = \frac{\ell_x}{\ell_0}, F_X(x) = \frac{\ell_0 - \ell_x}{\ell_0}, \ \mu(x) = -\frac{d}{dx}\log(\ell_x),$$
$${}_tp_x = \frac{\ell_{x+t}}{\ell_x}, \ {}_tq_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} = \frac{{}_td_x}{\ell_x}, p_x = \frac{\ell_{x+1}}{\ell_x}, \ q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} = \frac{d_x}{\ell_x}, \ {}_{n|m}q_x = \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x}.$$

$$\overset{\circ}{e}_0 = \int_0^\infty \frac{\ell_x}{\ell_0}dx, \overset{\circ}{e}_x = \int_0^\infty \frac{\ell_{x+t}}{\ell_x}dt, \ \overset{\circ}{e}_{x:\overline{n}|} = \int_0^n \frac{\ell_{x+t}}{\ell_x}dt, \ e_x = \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x}, \ e_{x:\overline{n}|} = \sum_{k=1}^{n} \frac{\ell_{x+k}}{\ell_x}.$$

The expected number of years lived between age x and age $x + n$ by the ℓ_x survivors at age x is ${}_nL_x$.

$${}_nL_x = \ell_x \overset{\circ}{e}_{x:\overline{n}|} = \int_0^n \ell_{x+t}\, dt, \ L_x = {}_1L_x = \ell_x \overset{\circ}{e}_{x:\overline{1}|}, \overset{\circ}{e}_x = \frac{\sum_{k=x}^{\infty} L_k}{\ell_x}, \ \overset{\circ}{e}_{x:\overline{n}|} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}.$$

Interpolation	ℓ_{x+t}	${}_tp_x$	L_x
uniform distribution of deaths	$\ell_x + t(\ell_{x+1} - \ell_x)$	$1 - tq_x$	$\frac{\ell_x + \ell_{x+1}}{2}$
exponential interpolation	$\ell_x p_x^t$	p_x^t	$\frac{d_x}{-\log p_x}$
Balducci assumption	$\frac{1}{(1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}}$	$\frac{p_x}{t + (1-t)p_x}$	$\frac{-\ell_{x+1}\log p_x}{q_x}$

Under uniform distribution of deaths:

$$\ell_{x+t} = \ell_x + t(\ell_{x+1} - \ell_x),\ {}_tp_x = 1 - tq_x,\ f_{T(x)}(t) = q_x, \mu_{x+t} = \frac{q_x}{1 - tq_x},\ 0 \le t \le 1,$$
$$L_x = \frac{\ell_x + \ell_{x+1}}{2},\ \mathring{e}_x = e_x + \frac{1}{2}.$$

Under exponential interpolation:

$$\ell_{x+t} = \ell_x p_x^t, {}_tp_x = p_x^t,\ f_{T_x}(t) = -p_x^t \log p_x,\ \mu_{x+t} = -\log p_x,\ 0 \le t \le 1.$$

Under (Balducci assumption) harmonic interpolation:

$$\frac{1}{\ell_{x+t}} = (1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}},\ {}_tp_x = \frac{p_x}{t + (1-t)p_x}.$$

4 Life insurance.

type of insurance	payment
whole life insurance	$Z_x = v^{K_x}$
n–year term life insurance	$Z^1_{x:\overline{n}\vert} = v^{K_x} I(K_x \le n)$
n–year deferred life insurance	${}_n\vert Z_x = v^{K_x} I(n < K_x)$
n–year pure endowment life insurance	$Z_{x:\overline{n}\vert}^{\ \ 1} = v^n I(n < K_x)$
n–year endowment life insurance	$Z_{x:\overline{n}\vert} = v^{\min(K_x,n)}$
m–year deferred n–year term life insurance	${}_m\vert_n Z_x = v^{K_x} I(m < K_x \le m+n)$

Whole life insurance paid at the end of the year:

$$Z_x = v^{K_x}, A_x = E[Z_x] = \sum_{k=1}^{\infty} v^k {}_{k-1}p_x \cdot q_{x+k-1},\ {}^2A_x = \sum_{k=1}^{\infty} v^{2k} {}_{k-1}p_x \cdot q_{x+k-1},$$
$$\mathrm{Var}(Z_x) = {}^2A_x - A_x^2,\ A_x = vq_x + vp_x A_{x+1}.$$

n**–year term life insurance paid at the end of the year:**

$$Z^1_{x:\overline{n}\vert} = v^{K_x} I(K_x \le n), A^1_{x:\overline{n}\vert} = E[Z^1_{x:\overline{n}\vert}] = \sum_{k=1}^{n} v^k \cdot {}_{k-1}\vert q_x,\ {}^2A^1_{x:\overline{n}\vert} = \sum_{k=1}^{n} v^{2k} \cdot {}_{k-1}\vert q_x,$$
$$\mathrm{Var}(Z^1_{x:\overline{n}\vert}) = {}^2A^1_{x:\overline{n}\vert} - {A^1_{x:\overline{n}\vert}}^2,\ A^1_{x:\overline{n}\vert} = vq_x + vp_x A^1_{x+1:\overline{n-1}\vert}.$$

n**–year deferred life insurance paid at the end of the year:**

$${}_n\vert Z_x = v^{K_x} I(n < K_x), {}_n\vert A_x = E[{}_n\vert Z_x] = \sum_{k=n+1}^{\infty} v^k \cdot {}_{k-1}\vert q_x,\ {}^2{}_n\vert A_x = \sum_{k=n+1}^{\infty} v^{2k} \cdot {}_{k-1}\vert q_x,$$
$$\mathrm{Var}({}_n\vert Z_x) = {}^2{}_n\vert A_x - {{}_n\vert A_x}^2,\ {}_n\vert A_x = v^n {}_{n-1}p_x \cdot q_{x+n-1} + {}_{n+1}\vert A_x.$$

n–year pure endowment life insurance paid at the end of the year:

$$Z_{x:\overline{n}|}^{\ \ 1} = v^n I(n < K_x), A_{x:\overline{n}|}^{\ \ 1} = E[Z_{x:\overline{n}|}^{\ \ 1}] = {}_nE_x = v^n \cdot {}_np_x,$$
$$^2A_{x:\overline{n}|}^{\ \ 1} = v^{2n} \cdot {}_np_x,\ \mathrm{Var}(Z_{x:\overline{n}|}^{\ \ 1}) = {}^2A_{x:\overline{n}|}^{\ \ 1} - {A_{x:\overline{n}|}^{\ \ 1}}^2.$$

n–year endowment life insurance paid at the end of the year:

$$Z_{x:\overline{n}|} = v^{\min(K_x,n)}, A_{x:\overline{n}|} = {}_nE_x = E[Z_{x:\overline{n}|}] = \sum_{k=1}^{n} v^k \cdot {}_{k-1}|q_x + v^n{}_np_x,$$
$$^2A_{x:\overline{n}|} = \sum_{k=1}^{n} v^{2k} \cdot {}_{k-1}|q_x + v^{2n}{}_np_x,\ \mathrm{Var}(Z_{x:\overline{n}|}) = {}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}^2.$$

$${}_n|A_x = {}_nE_xA_{x+n}, A_x = A^1_{x:\overline{n}|} + {}_n|A_x = A^1_{x:\overline{n}|} + {}_nE_xA_{x+n},\ {}^2A_x = {}^2A^1_{x:\overline{n}|} + {}^2{}_n|A_x,$$
$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1},\ {}^2A_{x:\overline{n}|} = {}^2A^1_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\ \ 1},$$

Increasing/decreasing life insurance paid at the end of the year:

$$(IA)_x = \sum_{k=1}^{\infty} kv^k \cdot {}_{k-1}|q_x, (IA)^1_{x:\overline{n}|} = \sum_{k=1}^{n} kv^k \cdot {}_{k-1}|q_x,\ (DA)^1_{x:\overline{n}|} = \sum_{k=1}^{n}(n+1-k)v^k \cdot {}_{k-1}|q_x.$$

Under de Moivre's model with terminal age ω, if ω, x, n are a positive integers,

$$A_x = \frac{a_{\overline{\omega-x}|i}}{\omega-x},\ A^1_{x:\overline{n}|} = \frac{a_{\overline{n}|}}{\omega-x},\ A_{x:\overline{n}|}^{\ \ 1} = v^n\frac{\omega-x-n}{\omega-x},\ {}_n|A_x = v^n\frac{a_{\overline{\omega-x-n}|i}}{\omega-x}.$$

Under constant force of mortality:

$$A_x = \frac{q_x}{q_x+i},\ A_{x:\overline{n}|}^{\ \ 1} = e^{-n(\mu+\delta)},\ {}_n|A_x = e^{-n(\mu+\delta)}\frac{q_x}{q_x+i}, A^1_{x:\overline{n}|} = (1-e^{-n(\mu+\delta)})\frac{q_x}{q_x+i}.$$

type of insurance	payment
whole life insurance	$\overline{Z}_x = v^{K_x}$
n–year term life insurance	$\overline{Z}^1_{x:\overline{n}|} = v^{T_x}I(T_x \le n)$
n–year deferred life insurance	${}_n|\overline{Z}_x = v^{T_x}I(n < T_x)$
n–year pure endowment life insurance	$\overline{Z}_{x:\overline{n}|}^{\ \ 1} = v^nI(n < T_x)$
n–year endowment life insurance	$\overline{Z}_{x:\overline{n}|} = v^{\min(T_x,n)}$
m–year deferred n-year term life insurance	${}_m|_n\overline{Z}_x = v^{T_x}I(m \le T_x \le m+n)$

Whole life insurance paid at the time of death:

$$\overline{Z}_x = v^{T_x}\overline{A}_x = E[\overline{Z}_x] = \int_0^\infty v^t f_{T_x}(t)\,dt,$$
$$^2\overline{A}_x = E[(\overline{Z}_x)^2] = \int_0^\infty v^{2t} f_{T_x}(t)\,dt,\ \mathrm{Var}(\overline{Z}_x) = {}^2\overline{A}_x - \overline{A}_x^2.$$

n–year term life insurance paid at the time of death:

$$\overline{Z}^{1}_{x:\overline{n}|} = v^{T_x} I(T_x \le n), \overline{A}^{1}_{x:\overline{n}|} = E[\overline{Z}^{1}_{x:\overline{n}|}] = \int_0^n v^t f_{T_x}(t)\, dt,$$
$$^2\overline{A}^{1}_{x:\overline{n}|} = E[\overline{Z}^{1}_{x:\overline{n}|}{}^2] = \int_0^n v^{2t} f_{T_x}(t)\, dt,\ \mathrm{Var}(\overline{Z}^{1}_{x:\overline{n}|}) = {}^2A^{1}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|}{}^2.$$

n–year deferred life insurance paid at the time of death:

$$_n|\overline{Z}_x = v^{T_x} I(n < T_x),\ {}_n|\overline{A}_x = E[_n|\overline{Z}_x] = \int_n^\infty v^t f_{T_x}(t)\, dt,$$
$$^2{}_n|\overline{A}_x = E[_n|\overline{Z}_x{}^2] = \int_n^\infty v^{2t} f_{T_x}(t)\, dt,\ \mathrm{Var}(_n|\overline{Z}_x) = {}^2{}_n|\overline{A}_x - {}_n|\overline{A}_x{}^2.$$

n–year endowment life insurance:

$$\overline{Z}_{x:\overline{n}|} = v^{\min(T_x,n)}, \overline{A}_{x:\overline{n}|} = E[\overline{Z}_{x:\overline{n}|}] = \int_0^n v^t f_{T_x}(t)\, dt + v^n \Pr\{T_x > n\},$$
$$^2\overline{A}_{x:\overline{n}|} = E[(\overline{Z}_{x:\overline{n}|})^2] = \int_0^n v^{2t} f_{T_x}(t)\, dt + v^{2n} \Pr\{T_x > n\},\ \mathrm{Var}(\overline{Z}_{x:\overline{n}|}) = {}^2\overline{A}_{x:\overline{n}|} - \overline{A}_{x:\overline{n}|}{}^2.$$

$$\overline{Z}_x = \overline{Z}^{1}_{x:\overline{n}|} + {}_n|\overline{Z}_x,\ \overline{A}_x = \overline{A}^{1}_{x:\overline{n}|} + {}_n|\overline{A}_x,\ {}^2\overline{A}_x = {}^2\overline{A}^{1}_{x:\overline{n}|} + {}^2{}_n|\overline{A}_x,$$
$$\overline{Z}_{x:\overline{n}|} = \overline{Z}^{1}_{x:\overline{n}|} + E_{x:\overline{n}|}^{\ \ 1},\ \overline{A}_{x:\overline{n}|} = \overline{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1},\ {}^2\overline{A}_{x:\overline{n}|} = {}^2\overline{A}^{1}_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\ \ 1},$$
$$_n|\overline{A}_x = {}_nE_x \overline{A}_{x+n}.$$

Under de Moivre's model with terminal age ω,

$$\overline{A}_x = \frac{\overline{a}_{\overline{\omega-x}|i}}{\omega - x},\ \overline{A}^{1}_{x:\overline{n}|} = \frac{\overline{a}_{\overline{n}|i}}{\omega - x},\ \overline{A}_{x:\overline{n}|}^{\ \ 1} = e^{-n\delta}\frac{\omega - x - n}{\omega - x},\ {}_n|\overline{A}_x = e^{-n\delta}\frac{\overline{a}_{\overline{\omega-x-n}|i}}{\omega - x}.$$

Under constant force of mortality:

$$\overline{A}_x = \frac{\mu}{\mu+\delta},\ \overline{A}_{x:\overline{n}|}^{\ \ 1} = e^{-n(\mu+\delta)},\ {}_n|\overline{A}_x = e^{-n(\mu+\delta)}\frac{\mu}{\mu+\delta},\ \overline{A}^{1}_{x:\overline{n}|} = (1 - e^{-n(\mu+\delta)})\frac{\mu}{\mu+\delta}.$$

Continuously increasing whole life insurance: $b_t = t$, $t \ge 0$,

$$\left(\overline{I}\ \overline{A}\right)_x = \int_0^\infty t v^t \cdot {}_tp_x \mu_{x+t}\, dt.$$

Annually increasing whole life insurance: $b_t = \lceil t \rceil$, $t \ge 0$, present value is denoted by

$$\left(I\ \overline{A}\right)_x = \sum_{k=1}^{\infty} \int_{k-1}^{k} k v^t \cdot {}_tp_x \mu_{x+t}\, dt.$$

n–year term continuously increasing whole life insurance: $b_t = t$, $0 \le t \le n$,

$$\left(\overline{I}\,\overline{A}\right)^{1}_{x:\overline{n}|} = \int_0^n t v^t \cdot {}_t p_x \mu_{x+t}\,dt.$$

n–year term annually increasing whole life insurance: $b_t = \lceil t \rceil$, $0 \le t \le n$,

$$\left(I\,\overline{A}\right)^{1}_{x:\overline{n}|} = \sum_{k=1}^{n} \int_{k-1}^{k} k v^t \cdot {}_t p_x \mu_{x+t}\,dt.$$

Continuously decreasing life insurance: $b_t = n - t$, $0 \le t \le n$,

$$\left(\overline{D}\,\overline{A}\right)^{1}_{x:\overline{n}|} = \int_0^n (n-t) v^t \cdot {}_t p_x \mu_{x+t}\,dt.$$

Annually decreasing life insurance: $b_t = \lceil n - t \rceil$, $0 \le t \le n$,

$$\left(D\,\overline{A}\right)^{1}_{x:\overline{n}|} = \sum_{k=1}^{n} \int_{k-1}^{k} (n+1-k) v^t \cdot {}_t p_x \mu_{x+t}\,dt.$$

Assuming a uniform distribution of deaths:

$$\overline{A}_x = \frac{i}{\delta} A_x,\ \overline{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|},\ {}_n|\overline{A}_x = \frac{i}{\delta} \cdot {}_n|A_x,\ \overline{A}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1},$$

$$A^{(m)}_x = \frac{i}{i^{(m)}} A_x,\ \overset{(m)}{A}{}^{1}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|},\ {}_n|A^{(m)}_x = \frac{i}{i^{(m)}} \cdot {}_n|A_x,\ A^{(m)}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}.$$

5 Life annuities.

due annuities	present value	APV
whole life	$\ddot{Y}_x = \ddot{a}_{\overline{K_x}|} = \frac{1-Z_x}{d}$	$\ddot{a}_x = \frac{1-A_x}{d}$
n–year deferred life insurance	${}_n|\ddot{Y}_x = v^n \ddot{a}_{\overline{K_x-n}|} I(K_x > n)$	${}_n|\ddot{a}_x = {}_nE_x \ddot{a}_{x+n}$
n–year term	$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_n,n)}|} = \frac{1-Z_{x:\overline{n}|}}{d}$	$\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$

immediate annuities	present value	APV
whole life	$Y_x = a_{\overline{K_x-1}|} = \frac{v-Z_x}{d}$	$a_x = \frac{v-A_x}{d}$
n–year deferred life insurance	${}_n|Y_x = v^n a_{\overline{K_x-n-1}|} I(K_x > n+1)$	${}_n|a_x = {}_nE_x \cdot a_{x+n}$
n–year term	$Y_{x:\overline{n}|} = a_{\overline{\min(K_x-1,n)}|} = \frac{v-Z_{x:\overline{n+1}|}}{d}$	$a_{x:\overline{n}|} = \frac{v-A_{x:\overline{n+1}|}}{d}$

continuous annuities	present value	APV
whole life	$\overline{Y}_x = \overline{a}_{\overline{T_x}|} = \frac{1-\overline{Z}_x}{\delta}$	$\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$
n–year deferred life insurance	${}_n|\overline{Y}_x = v^n \overline{a}_{\overline{T_x-n}|} I(T_x > n)$	${}_n|\overline{a}_x = {}_nE_x \cdot \overline{a}_{x+n}$
n–year term	$\overline{Y}_{x:\overline{n}|} = \overline{a}_{\overline{\min(T_x,n)}|} = \frac{1-v^{\min(T_x,n)}}{\delta}$	$\overline{a}_{x:\overline{n}|} = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$

Discrete whole life due annuity:

$$\ddot{Y}_x = \ddot{a}_{\overline{K_x}|} = \frac{1-Z_x}{d},\ \ddot{a}_x = \frac{1-A_x}{d} = \sum_{k=0}^{\infty} v^k {}_kp_x,\ \mathrm{Var}(\ddot{Y}_x) = \frac{{}^2A_x - A_x^2}{d^2},\ \ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}.$$

Whole life immediate annuity:

$$Y_x = a_{\overline{K_x-1}|} = \ddot{Y}_x - 1 = \frac{v-Z_x}{d},\ a_x = \frac{v-A_x}{d} = \sum_{k=1}^{\infty} v^k {}_kp_x,$$

$$\mathrm{Var}(Y_x) = \frac{{}^2A_x - A_x^2}{d^2},\ a_x = vp_x\ddot{a}_{x+1} = vp_x(1+a_{x+1}).$$

Whole life continuous annuity:

$$\overline{Y}_x = \overline{a}_{\overline{T_x}|} = \frac{1-\overline{Z}_x}{\delta},\ \overline{a}_x = \frac{1-\overline{A}_x}{\delta} = \int_0^{\infty} v^t \cdot {}_tp_x\,dt,\ \mathrm{Var}(\overline{Y}_x) = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2}.$$

n–**year deferred discrete due annuity:**

$${}_{n}|\ddot{Y}_x = v^n\ddot{a}_{\overline{K_x-n}|}I(K_x > n),\ {}_{n}|\ddot{a}_x = \sum_{k=n}^{\infty} v^k \cdot {}_kp_x = {}_nE_x\ddot{a}_{x+n}.$$

n–**year deferred discrete immediate annuity:**

$${}_{n}|Y_x = {}_{n+1}|\ddot{Y}_x,\ {}_{n}|a_x = {}_{n+1}|\ddot{a}_x = vp_x \cdot {}_{n-1}|a_{x+1}.$$

n–**year deferred continuous annuity:**

$${}_{n}|\overline{Y}_x = v^n\overline{a}_{\overline{T_x-n}|}I(T_x > n),\ {}_{n}|\overline{a}_x = \int_n^{\infty} v^t \cdot {}_tp_x\,dt = {}_nE_x \cdot \overline{a}_{x+n}.$$

n–**year term due discrete annuity:**

$$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_n,n)}|} = \frac{1-Z_{x:\overline{n}|}}{d},\ \ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x = \frac{1-A_{x:\overline{n}|}}{d},$$

$$\mathrm{Var}(\ddot{Y}_{x:\overline{n}|}) = \frac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2},\ \ddot{a}_{x:\overline{n+m}|} = \ddot{a}_{x:\overline{n}|} + {}_nE_x \cdot \ddot{a}_{x+n:\overline{m}|},$$

$$\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n}|\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x\ddot{a}_{x+n}.$$

n–**year term discrete immediate annuity:**

$$Y_{x:\overline{n}|} = a_{\overline{\min(K_x-1,n)}|} = \ddot{Y}_{x:\overline{n+1}|} - 1 = \frac{v-Z_{x:\overline{n+1}|}}{d},$$

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n+1}|} - 1 = \sum_{k=1}^{n} v^k \cdot {}_kp_x = \frac{v-A_{x:\overline{n+1}|}}{d}$$

$$\mathrm{Var}(Y_{x:\overline{n}|}) = \frac{{}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2}{d^2},$$

$$a_x = {}_{n}|a_x + a_{x:\overline{n}|} = {}_{n}|a_x + {}_nE_xa_{x+n},$$

n–year term continuous annuity:

$$\overline{Y}_{x:\overline{n}|} = \overline{a}_{\overline{\min(T_x,n)}|} = \frac{1-v^{\min(T_x,n)}}{\delta} = \frac{1-\overline{Z}_{x:\overline{n}|}}{\delta},\ \overline{a}_{x:\overline{n}|} = \int_0^n v^s\,{}_sp_x\,ds = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta},$$

$$\mathrm{Var}(\overline{Y}_{x:\overline{n}|}) = \frac{{}^2\overline{A}_{x:\overline{n}|}-(\overline{A}_{x:\overline{n}|})^2}{\delta^2},\ \overline{a}_{x:\overline{n+m}|} = \overline{a}_{x:\overline{n}|} + {}_nE_x\cdot\overline{a}_{x+n:\overline{m}|},\ \overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_n|\overline{a}_x.$$

Under constant force of mortality:

$$\ddot{a}_x = \frac{1}{1-vp_x} = \frac{1+i}{q_x+i} = \frac{1}{1-e^{-(\delta+\mu)}},\ a_x = \frac{vp_x}{1-vp_x} = \frac{1-q_x}{q_x+i} = \frac{e^{-(\delta+\mu)}}{1-e^{-(\delta+\mu)}},\ \overline{a}_x = \frac{1}{\mu+\delta}.$$

Annuities paid m times a year.

For a whole life unity annuity–due to (x) paid m times a year:

$$\ddot{Y}_x^{(m)} = \frac{1-Z_x^{(m)}}{d^{(m)}},\ \ddot{a}_x^{(m)} = \frac{1-A_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{\infty} v^{\frac{k}{m}}\cdot{}_{\frac{k}{m}}p_x,\ \mathrm{Var}(\ddot{Y}_x^{(m)}) = \frac{{}^2A_x^{(m)}-(A_x^{(m)})^2}{(d^{(m)})^2}.$$

For a whole life unity annuity–immediate to (x) paid m times a year:

$$Y_x^{(m)} = \ddot{Y}_x^{(m)} - \frac{1}{m} = \frac{v^{1/m}-Z_x^{(m)}}{d^{(m)}},$$

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} = \frac{v^{1/m}-A_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=1}^{\infty} v^{\frac{k}{m}}\cdot{}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(Y_x^{(m)}) = \frac{{}^2A_x^{(m)}-(A_x^{(m)})^2}{(d^{(m)})^2}.$$

For a n–year unity annuity–due to (x) paid m times a year:

$$\ddot{Y}_{x:\overline{n}|}^{(m)} = \frac{1-Z_{x:\overline{n}|}^{(m)}}{d^{(m)}},\ \ddot{a}_{x:\overline{n}|}^{(m)} = \frac{1-A_{x:\overline{n}|}^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{nm-1} v^{\frac{1}{m}}\cdot{}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(\ddot{Y}_{x:\overline{n}|}^{(m)}) = \frac{\mathrm{Var}(Z_{x:\overline{n}|}^{(m)})}{(d^{(m)})^2}.$$

For a n–year unity annuity–due to (x) paid m times a year:

$$\ddot{Y}_{x:\overline{n}|}^{(m)} = \frac{1-Z_{x:\overline{n}|}^{(m)}}{d^{(m)}},\ \ddot{a}_{x:\overline{n}|}^{(m)} = \frac{1-A_{x:\overline{n}|}^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{nm-1} v^{\frac{1}{m}}\cdot{}_{\frac{k}{m}}p_x,\ \mathrm{Var}(\ddot{Y}_{x:\overline{n}|}^{(m)}) = \frac{\mathrm{Var}(Z_{x:\overline{n}|}^{(m)})}{(d^{(m)})^2}.$$

For a n–year unity annuity–immediate to (x) paid m times a year:

$$Y_{x:\overline{n}|}^{(m)} = \ddot{Y}_{x:\overline{n}|}^{(m)} - \frac{1}{m} + \frac{1}{m}Z_{x:\overline{\frac{1}{n}}|},\ a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m} + \frac{1}{m}\cdot{}_nE_x.$$

For a n–year deferred unity annuity–due to (x) paid m times a year:

$$
{}_{n|}\ddot{Y}_x^{(m)} = \frac{Z_{x:\overline{n}|}^{\;\;1} - {}_{n|}Z_x^{(m)}}{d^{(m)}}, \; {}_{n|}\ddot{a}_x^{(m)} = \frac{A_{x:\overline{n}|}^{\;\;1} - {}_{n|}A_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=nm}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x = {}_nE_x \cdot \ddot{a}_{x+n}^{(m)},
$$

$$
\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + {}_{n|}\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + {}_nE_x\ddot{a}_{x+n}^{(m)}.
$$

For a n–year deferred unity annuity–immediate to (x) paid m times a year:

$$
{}_{n|}Y_x^{(m)} = {}_{n|}\ddot{Y}_x^{(m)} - \frac{1}{m}Z_{x:\overline{n}|}^{\;\;1}, \; {}_{n|}a_x^{(m)} = {}_nE_x \cdot a_{x+n}^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x,
$$

$$
a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_{n|}a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_nE_x a_{x+n}^{(m)}.
$$

Under an uniform distribution of deaths within each year:

$$
\ddot{a}_x^{(m)} = \frac{1 - \frac{i}{i^{(m)}}A_x}{d^{(m)}}, \; a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} = \frac{v^{1/m} - \frac{i}{i^{(m)}}A_x}{d^{(m)}}, \; \overline{a}_x = \frac{1 - \frac{i}{\delta}A_x}{\delta}.
$$

6 Benefit Premiums.

Fully discrete insurance

Whole life insurance:

$$
L_x = v^{K_x} - P\ddot{a}_{\overline{K_x}|} = Z_x - P\ddot{Y}_x = Z_x - P\ddot{Y}_x = Z_x\left(1 + \frac{P}{d}\right) - \frac{P}{d},
$$

$$
E[L_x] = A_x - P\ddot{a}_x = A_x\left(1 + \frac{P}{d}\right) - \frac{P}{d},
$$

$$
\mathrm{Var}(L_x) = \left(1 + \frac{P}{d}\right)^2 \mathrm{Var}(Z_x) = \left(1 + \frac{P}{d}\right)^2 \left({}^2A_x - {A_x}^2\right).
$$

Under the equivalence principle:

$$
P_x = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1 - A_x} = \frac{1}{\ddot{a}_x} - d,
$$

$$
\mathrm{Var}(L_x) = \frac{{}^2A_x - {A_x}^2}{(1 - A_x)^2} = \frac{{}^2A_x - {A_x}^2}{(d\ddot{a}_x)^2}, \; {}_tP_x = \frac{A_x}{\ddot{a}_{x:\overline{t}|}}.
$$

n–year term insurance:

$$
L_{x:\overline{n}|}^1 = Z_{x:\overline{n}|}^1 - P\ddot{Y}_{x:\overline{n}|} = Z_{x:\overline{n}|}^1 - P\frac{1 - Z_{x:\overline{n}|}}{d},
$$

$$
P_{x:\overline{n}|}^1 = P(A_{x:\overline{n}|}^1) = \frac{A_{x:\overline{n}|}^1}{\ddot{a}_{x:\overline{n}|}}, \; {}_tP_{x:\overline{n}|}^1 = P({}_tA_{x:\overline{n}|}^1) = \frac{A_{x:\overline{n}|}^1}{\ddot{a}_{x:\overline{t}|}}.
$$

n–year pure endowment:

$$L_{x:\overset{\ 1}{\overline{n}|}} = Z_{x:\overset{\ 1}{\overline{n}|}} - P\ddot{Y}_{x:\overline{n}|} = Z_{x:\overset{\ 1}{\overline{n}|}} - P\frac{1 - Z_{x:\overline{n}|}}{d},$$

$$P_{x:\overset{\ 1}{\overline{n}|}} = P(A_{x:\overset{\ 1}{\overline{n}|}}) = \frac{A_{x:\overset{\ 1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}},\ {}_tP_{x:\overset{\ 1}{\overline{n}|}} = P({}_tA_{x:\overset{\ 1}{\overline{n}|}}) = \frac{A_{x:\overset{\ 1}{\overline{n}|}}}{\ddot{a}_{x:\overline{t}|}}.$$

n–year endowment:

$$L_{x:\overline{n}|} = v^{\min(n,K_x)} - P\ddot{a}_{\overline{\min(K_x,n)}|} = Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|} = Z_{x:\overline{n}|} - P\frac{1 - Z_{x:\overline{n}|}}{d} = \left(1 + \frac{P}{d}\right) Z_{x:\overline{n}|} - \frac{P}{d},$$

$$\mathrm{Var}(L_{x:\overline{n}|}) = \left(1 + \frac{P}{d}\right)^2 \mathrm{Var}(Z_{x:\overline{n}|}) = \left(1 + \frac{P}{d}\right)^2 \left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right),$$

$$P_{x:\overline{n}|} = P(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}, {}_tP_{x:\overline{n}|} = P({}_tA_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}},$$

$$\mathrm{Var}(L_{x:\overline{n}|}) = \left(1 + \frac{P_{x:\overline{n}|}}{d}\right)^2 \left({}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}{}^2\right) = \frac{{}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}{}^2}{\left(1 - A_{x:\overline{n}|}\right)^2} = \frac{{}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}{}^2}{\left(d\ddot{a}_{x:\overline{n}|}\right)^2}.$$

n–year deferred insurance:

$${}_n|Z_x - P\ddot{Y}_x,\ P({}_n|A_x) = \frac{{}_n|A_x}{\ddot{a}_x},\ {}_tP({}_n|A_x) = \frac{{}_n|A_x}{\ddot{a}_{x:\overline{t}|}}.$$

Properties:

$$P_{x:\overline{n}|} = P^1_{x:\overline{n}|} + P_{x:\overset{\ 1}{\overline{n}|}},\ {}_nP_x = P^1_{x:\overline{n}|} + P_{x:\overset{\ 1}{\overline{n}|}}A_{x+n}.$$

Semicontinuous annual benefit premiums

Whole life insurance:

$$\overline{P}_x = \overline{P}(A_x) = \frac{A_x}{\overline{a}_x},\ {}_t\overline{P}_x = {}_t\overline{P}(A_x) = \frac{A_x}{\overline{a}_{x:\overline{t}|}}.$$

n–year term insurance:

$$\overline{P}^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}},\ {}_t\overline{P}^1_{x:\overline{n}|} = {}_t\overline{P}(A^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\overline{a}_{x:\overline{t}|}}$$

n–year pure endowment:

$$\overline{P}_{x:\overset{\ 1}{\overline{n}|}} = \overline{P}(A_{x:\overset{\ 1}{\overline{n}|}}) = \frac{A_{x:\overset{\ 1}{\overline{n}|}}}{\overline{a}_{x:\overline{n}|}},\ {}_t\overline{P}_{x:\overset{\ 1}{\overline{n}|}} = {}_t\overline{P}(A_{x:\overset{\ 1}{\overline{n}|}}) = \frac{A_{x:\overset{\ 1}{\overline{n}|}}}{\overline{a}_{x:\overline{t}|}}.$$

n–year endowment:

$$\overline{P}_{x:\overline{n}|} = \overline{P}(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}},\ {}_t\overline{P}_{x:\overline{n}|} = {}_t\overline{P}(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\overline{a}_{x:\overline{t}|}}.$$

n–year deferred insurance:

$$\overline{P}({}_n|A_x) = \frac{{}_n|A_x}{\overline{a}_{x:\overline{n}|}},\ {}_t\overline{P}({}_n|A_x) = \frac{{}_n|A_x}{\overline{a}_{x:\overline{n}|}}$$

Fully continuous insurance

Whole life insurance:

$$L(\overline{A}_x) = v^{T_x} - P\overline{a}_{\overline{T_x}|} = \overline{Z}_x - P\overline{Y}_x = \overline{Z}_x - \frac{1-\overline{Z}_x}{\delta} = \overline{Z}_x\left(1+\frac{P}{\delta}\right) - \frac{P}{\delta},$$

$$\mathrm{Var}(L(\overline{A}_x)) = \left(1+\frac{P}{\delta}\right)^2 \mathrm{Var}(\overline{Z}_x) = \left(1+\frac{P}{\delta}\right)^2 \left({}^2\overline{A}_x - \overline{A}_x{}^2\right),$$

$$\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x} = \frac{\delta\overline{A}_x}{1-\overline{A}_x} = \frac{1}{\overline{a}_x} - \delta,\ {}_t\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_{x:\overline{t}|}}.$$

$$\mathrm{Var}(L(\overline{A}_x)) = \left(1+\frac{\overline{P}(\overline{A}_x)}{\delta}\right)^2 \left({}^2\overline{A}_x - \overline{A}_x{}^2\right) = \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(1-\overline{A}_x)^2} = \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(\delta\overline{a}_x)^2}.$$

n–year term insurance:

$$L = \overline{Z}^1_{x:\overline{n}|} - P\overline{Y}_{x:\overline{n}|},\ \overline{P}(\overline{A}^1_{x:\overline{n}|}) = \frac{\overline{A}^1_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}},\ {}_t\overline{P}(\overline{A}^1_{x:\overline{n}|}) = \frac{\overline{A}^1_{x:\overline{n}|}}{\overline{a}_{x:\overline{t}|}}.$$

n–year pure endowment:

$$L = \overline{Z}_{x:\overline{n}|}^{\ \ 1} - P\overline{Y}_{x:\overline{n}|},\ \overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ 1}) = \frac{\overline{A}_{x:\overline{n}|}^{\ \ 1}}{\overline{a}_{x:\overline{n}|}},\ {}_t\overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ 1}) = \frac{\overline{A}_{x:\overline{n}|}^{\ \ 1}}{\overline{a}_{x:\overline{t}|}}.$$

n–year endowment:

$$L = \overline{Z}_{x:\overline{n}|} - P\overline{Y}_{x:\overline{n}|} = \overline{Z}_{x:\overline{n}|} - P\frac{1-\overline{Z}_{x:\overline{n}|}}{\delta} = \left(1+\frac{P}{\delta}\right) - \frac{P}{\delta}\overline{Z}_{x:\overline{n}|},$$

$$\mathrm{Var}(L) = \left(1+\frac{P}{\delta}\right)\left({}^2\overline{A}_{x:\overline{n}|} - \left(\overline{A}_{x:\overline{n}|}\right)^2\right),$$

$$\overline{P}(\overline{A}_{x:\overline{n}|}) = \frac{\overline{A}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} = \frac{1-\delta\overline{a}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} = \frac{\delta\overline{A}_{x:\overline{n}|}}{1-\overline{A}_{x:\overline{n}|}},\ \mathrm{Var}(L) = \frac{{}^2\overline{A}_{x:\overline{n}|} - \overline{A}_{x:\overline{n}|}{}^2}{\left(1-\overline{A}_{x:\overline{n}|}\right)^2},\ {}_t\overline{P}(\overline{A}_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\overline{a}_{x:\overline{t}|}}.$$

n–year deferred insurance:

$$L = {}_n|\overline{Z}_x - P\overline{Y}_{x:\overline{n}|},\ \overline{P}({}_n|\overline{A}_x) = \frac{{}_n|\overline{A}_x}{\overline{a}_{x:\overline{n}|}}.$$

n–year deferred annuities

n–year deferred due annuity:

$$L = {}_{n}|\ddot{Y}_x - P\ddot{Y}_{x:\overline{n}|}, \ P({}_{n}|\ddot{a}_x) = \frac{{}_{n}|\ddot{a}_x}{\ddot{a}_{x:\overline{n}|}}.$$

n–year deferred immediate annuity:

$$L = {}_{n}|Y_x - P\ddot{Y}_{x:\overline{n}|}, \ P({}_{n}|a_x) = \frac{{}_{n}|a_x}{\ddot{a}_{x:\overline{n}|}}.$$

n–year deferred continuous annuity funded discretely:

$$L = {}_{n}|\overline{Y}_x - P\ddot{Y}_{x:\overline{n}|}, \ P({}_{n}|\overline{a}_x) = \frac{{}_{n}|\overline{a}_x}{\ddot{a}_{x:\overline{n}|}}.$$

n–year deferred continuous annuity funded continuously:

$$L = {}_{n}|\overline{Y}_x - P\overline{Y}_{x:\overline{n}|}, \ \overline{P}({}_{n}|\overline{a}_x) = \frac{{}_{n}|\overline{a}_x}{\ddot{a}_{x:\overline{n}|}}.$$