Manual for SOA Exam MLC.

Chapter 10. Markov chains.

Section 10.2. Markov chains.

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Markov chains

Definition 1

A **discrete time Markov chain** $\{X_n : n = 0, 1, 2, \dots\}$ *is a stochastic process with values in the countable space* E *such that for each* $i_0, i_1, \dots, i_n, j \in E$,

$$\begin{aligned} & \mathbb{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n\} \quad (1) \\ = & \mathbb{P}\{X_{n+1} = j | X_n = i_n\}. \end{aligned}$$

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$$= \mathbb{P}\{X_{n+1} = j | X_n = i_n\}.$$

Since X_n takes values in the countable set E, X_n has a discrete distribution.
The set E in the previous definition is called the **state space**.
Usually, $E = \{0, 1, 2, \dots\}$ or $E = \{1, 2, \dots, m\}$. We will assume that $E = \{0, 1, 2, \dots\}$. Each element of E is called a **state**. If $X_n = k$, where $k \in E$, we say that the Markov chain $\{X_n\}_{n=0}^{\infty}$ is at state k at stage n.

For a Markov chain the conditional distribution of any future state X_{n+1} given the past states $X_0, X_1, \dots, X_{n-1}$ and the present state X_n is independent of the past values and depends only on the present state. Having observed the process until time n, the distribution of the process after time n on depends only on the value of the process at time n. The interpretation of

$$\begin{aligned}&\mathbb{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n\}\\ =&\mathbb{P}\{X_{n+1} = j | X_n = i_n\}.\end{aligned}$$

is that given the present the future is independent of the past. In order words, the evolution of the process depends only on the present and not in the past.

Definition 2

Given events A, B and C such that $\mathbb{P}\{C\} > 0$, we say that A and B are independent given C if

$$\mathbb{P}\{A \cap B | C\} = \mathbb{P}\{A|C\}\mathbb{P}\{B|C\}.$$

Theorem 1

Given events A, B and C such that $\mathbb{P}\{B \cap C\} > 0$, we have that A and B are independent given C if and only if

$$\mathbb{P}\{A|B \cap C\} = \mathbb{P}\{A|C\}.$$

Proof: A and B are independent given C if and only if $\frac{\mathbb{P}\{A\cap B\cap C\}}{\mathbb{P}\{C\}} = \frac{\mathbb{P}\{A\cap C\}}{\mathbb{P}(C\}} \frac{\mathbb{P}\{B\cap C\}}{\mathbb{P}\{C\}}$. $\mathbb{P}\{A|B \cap C\} = \mathbb{P}(A|C\}$ if and only if $\frac{\mathbb{P}\{A\cap B\cap C\}}{\mathbb{P}\{B\cap C\}} = \frac{\mathbb{P}\{A\cap C\}}{\mathbb{P}\{C\}}$.

By the previous theorem, for a Markov chain $\{X_n : n = 0, 1, 2, \dots\}$, for each $i_0, \dots, i_{n+1} \in E$, given that $X_n = i_n$, $(X_0, X_1, \dots, X_{n-1}) = (i_0, \dots, i_{n-1})$ and $X_{n+1} = i_{n+1}$ are independent.

Example 1

A fair coin is thrown out repeatedly. Let X_n be the total number of heads obtained in the first n throws, $n = 0, 1, 2 \dots$. Notice that $X_0 = 0$. X_n has a binomial distribution with parameters n and $\frac{1}{2}$. The state space is $E = \{0, 1, 2, \dots, \}$. $\{X_n\}_{n=0}^{\infty}$ is a Markov chain because for each $i_0, i_1, \dots, i_n, j \in E$,

$$\begin{aligned} &\mathbb{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n\} \\ =&\mathbb{P}\{X_{n+1} = j | X_n = i_n\}, \\ =&\begin{cases} \frac{1}{2} & \text{if } j = i_n, \\ \frac{1}{2} & \text{if } j = i_n + 1, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Let $\alpha_n(i) = \mathbb{P}\{X_n = i\}$, $i \in E$.
$(\alpha_0(i))_{i\in E}$ is called the **initial distribution** of the Markov chain.
$(\alpha_n(i))_{i\in E}$ is called the distribution of the Markov chain at time n.
Notice that $\alpha_n(i) \geq 0$ and $\sum_{i\in E} \alpha_n(i) = 1$.
We will denote to the row vector $(\alpha_n(i))_{i\in E}$ by α_n. For example, if $E = \{0, 1, \dots, k\}$,

$$\alpha_n = (\alpha_n(0), \alpha_n(1), \dots, \alpha_n(k)).$$

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$$\alpha_n(i) \geq 0 \text{ and } \sum_{i\in E} \alpha_n(i) = 1.$$

Proof: $\alpha_n(i) = \mathbb{P}\{X_n = i\} \geq 0$. We also have that

$$\sum_{i\in E} \alpha_n(i) = \sum_{i\in E} \mathbb{P}\{X_n = i\} = \mathbb{P}\{X_n \in E\} = 1.$$

Let $Q_n(i,j) = \mathbb{P}\{X_{n+1} = j | X_n = i\}$, where $i, j \in E$. $Q_n(i,j)$ is called the **one–step transition probability** from state i into state j at stage n. We have that $(Q_n(i,j))_{i,j\in E}$ is a matrix. If $E = \{0, 1, , \dots, k\}$, then

$$(Q_n(i,j))_{i,j\in E} = \begin{pmatrix} Q_n(0,0) & Q_n(0,1) & \cdots & Q_n(0,k) \\ Q_n(1,0) & Q_n(1,1) & \cdots & Q_n(1,k) \\ \dots & \dots & \dots & \dots \\ Q_n(k,0) & Q_n(k,1) & \cdots & Q_n(k,k) \end{pmatrix}$$

The row of the matrix Q_n for the i entry is formed by conditional probabilities given $X_n = i$, the departing state. The column of the matrix Q_n for the j entry consists of conditional probabilities for $X_n = j$, the arriving state. If we write the states in the matrix Q_n, we have

$$\begin{array}{cc} & \text{arriving states} \\ & \begin{array}{cccc} 0 & 1 & \cdots & k \end{array} \\ \text{departing states} & \begin{array}{c} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ k \end{array} \begin{pmatrix} Q_n(0,0) & Q_n(0,1) & \cdots & Q_n(0,k) \\ Q_n(1,0) & Q_n(1,1) & \cdots & Q_n(1,k) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ Q_n(k,0) & Q_n(k,1) & \cdot & Q_n(k,k) \end{pmatrix} \end{array}$$

To find $Q_n(i,j)$ in this matrix, we need to look for i in the rows and j in the columns.

Example 2

Consider a Markov chain with $E = \{0, 1, 2\}$ *and*

$$Q_6 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}.$$

Find:
(i) $\mathbb{P}\{X_7 = j | X_6 = 0\}$, $j = 0, 1, 2$.
(ii) $\mathbb{P}\{X_7 = 1 | X_6 = j\}$, $j = 0, 1, 2$.

Solution: (i) We have that

$$\mathbb{P}\{X_7 = 0 | X_6 = 0\} = 0.2, \mathbb{P}\{X_7 = 1 | X_6 = 0\} = 0.3 \text{ and } \mathbb{P}\{X_7 = 2 | X_6 = 0\} = 0.5.$$

(ii) We have that

$$\mathbb{P}\{X_7 = 1 | X_6 = 0\} = 0.3, \mathbb{P}\{X_7 = 1 | X_6 = 1\} = 0.5 \text{ and } \mathbb{P}\{X_7 = 1 | X_6 = 2\} = 0.1.$$

In the previous example, $\mathbb{P}\{X_7 = j | X_6 = 0\}$, $j = 0, 1, 2$, are all the transition probabilities from state 0 at time 6 to another state at time 7. In the previous problem, $\mathbb{P}\{X_7 = j | X_6 = 0\}$, $j = 0, 1, 2$, are all the transition probabilities from time 6 to time 7 arriving at state 1.

Theorem 3

The one–step transition probabilities $Q_n(i,j)$ satisfy that

$$Q_n(i,j) \geq 0 \text{ and } \sum_{j \in E} Q_n(i,j) = 1,$$

i.e. the sum of the elements in each row is one.

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i.e. the sum of the elements in each row is one.

Proof.

Since $Q_n(i,j)$ is a conditional probability, $Q_n(i,j) \geq 0$. We also have that

$$\sum_{j \in E} Q_n(i,j) = \sum_{j \in E} \mathbb{P}\{X_{n+1} = j | X_n = i\} = \mathbb{P}\{X_{n+1} \in E | X_n = i\} = 1.$$

□

Example 3

Which of the following are legitime one-step transition probability matrices

(i) $Q_0 = \begin{pmatrix} -1 & 2 \\ 0.5 & 0.5 \end{pmatrix}$.

(ii) $Q_0 = \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.7 \end{pmatrix}$.

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(ii) $Q_0 = \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.7 \end{pmatrix}$.

Solution: (i) The matrix is not a legitime transition probability because the entry $(1, 1)$ is negative.

(ii) The matrix is not a legitime transition probability because the elements of the second row do not add to one.

We define $Q_n^{(k)}(i,j) = P\{X_{n+k} = j | X_n = i\}$. $Q_n^{(k)}(i,j)$ is called the k–step transition probability from state i into state j at time n.

Theorem 4

$$Q_n^{(k)}(i,j) \geq 0 \text{ and } \sum_{j \in E} Q_n^{(k)}(i,j) = 1.$$

Lemma 1

Successive conditioning rule. *For each* $B, A_1, A_2, \dots, A_n \subset \Omega$,

$$\begin{aligned}&\mathbb{P}\{A_1 \cap A_2 \cap \cdots \cap A_n | B\}\\ =&\mathbb{P}\{A_1|B\}\mathbb{P}\{A_2 \mid B \cap A_1\} \cdots \mathbb{P}\{A_n \mid B \cap A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}.\end{aligned}$$

Lemma 1

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Proof:

$$\begin{aligned}&\mathbb{P}\{A_1|B\}\mathbb{P}\{A_2 \mid B \cap A_1\} \cdots \mathbb{P}\{A_n \mid B \cap A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}\\ =&\frac{\mathbb{P}\{B \cap A_1\}}{\mathbb{P}\{B\}} \frac{\mathbb{P}\{B \cap A_1 \cap A_2\}}{\mathbb{P}\{B \cap A_1\}} \cdots \frac{\mathbb{P}\{B \cap A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap A_n\}}{\mathbb{P}\{B \cap A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}}\\ =&\frac{\mathbb{P}\{B \cap A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap A_n\}}{\mathbb{P}\{B\}}\\ =&\mathbb{P}\{A_1 \cap A_2 \cap \cdots \cap A_n | B\}.\end{aligned}$$

Corollary 1

For each $A_1, A_2, \dots, A_n \subset \Omega$,

$$\begin{aligned}&\mathbb{P}\{A_1 \cap A_2 \cap \cdots \cap A_n\}\\ =&\mathbb{P}\{A_1\}\mathbb{P}\{A_2 \mid A_1\} \cdots \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}.\end{aligned}$$

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Proof: By Lemma 1,

$$\begin{aligned}&\mathbb{P}\{A_1 \cap A_2 \cap \cdots \cap A_n|B\}\\=&\mathbb{P}\{A_1|B\}\mathbb{P}\{A_2 \mid B \cap A_1\} \cdots \mathbb{P}\{A_n \mid B \cap A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}.\end{aligned}$$

Taking $B = \Omega$, we get

$$\begin{aligned}&\mathbb{P}\{A_1 \cap A_2 \cap \cdots \cap A_n\}\\=&\mathbb{P}\{A_1\}\mathbb{P}\{A_2 \mid A_1\} \cdots \mathbb{P}\{A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}\}.\end{aligned}$$

Example 1

From a deck of 52 cards, you withdraw three cards one after another. Find the probability that the first two cards are spades and the third one is a club.

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From a deck of 52 cards, you withdraw three cards one after another. Find the probability that the first two cards are spades and the third one is a club.

Solution: Let $A_1 = \{\text{first card is a spade}\}$, let $A_2 = \{\text{second card is a spade}\}$ and let $A_3 = \{\text{third card is a club}\}$. We have that

$$\mathbb{P}\{A_1 \cap A_2 \cap A_3\} = \mathbb{P}\{A_1\}\mathbb{P}\{A_2|A_1\}\mathbb{P}\{A_3|A_1 \cap A_2\}$$
$$= \frac{13}{52}\frac{12}{51}\frac{13}{50} = 0.01529411765.$$

Theorem 5

(Basic theorem for Markov chains)

(i)

$$\begin{aligned} & P\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i_n\} \\ =& \alpha_0(i_0) Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n) \end{aligned}$$

(ii)

$$\begin{aligned} & \alpha_n(i_n) = P\{X_n = i_n\} \\ = & \sum_{i_0, i_1, \ldots, i_{n-1} \in E} \alpha_0(i_0) Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n). \end{aligned}$$

Proof: (i) Using Corollary 1 and the Markov property,

$$\begin{aligned}
&\mathbb{P}\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i_n\}\\
=&\mathbb{P}\{X_0 = i_0\}\mathbb{P}\{X_1 = i_1 | X_0 = i_0\}\mathbb{P}\{X_2 = i_2 | X_0 = i_0, X_1 = i_1\}\\
&\cdots \mathbb{P}\{X_n = i_n | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\}\\
=&\mathbb{P}\{X_0 = i_0\}\mathbb{P}\{X_1 = i_1 | X_0 = i_0\}\mathbb{P}\{X_2 = i_2 | X_1 = i_1\}\\
&\cdots \mathbb{P}\{X_n = i_n | X_{n-1} = i_{n-1}\}\\
=&\alpha_0(i_0) Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n).
\end{aligned}$$

(ii) Notice that

$$\{X_n = i_n\}$$
$$= \cup_{i_0,i_1,\ldots,i_{n-1}\in E} \{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i_n\},$$

where the union is over disjoint events. Hence,

$$\alpha_n(i_n) = \mathbb{P}\{X_n = i_n\}$$
$$= \sum_{i_0,i_1,\ldots,i_{n-1}\in E} \mathbb{P}\{X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i_n\}$$
$$= \sum_{i_0,i_1,\ldots,i_{n-1}\in E} \alpha_0(i_0) Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n).$$

$$\alpha_n(i_n) = \sum_{i_0, i_1, \ldots, i_{n-1} \in E} \alpha_0(i_0) Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n)$$

implies that

$$\alpha_n = \alpha_0 Q_0 Q_1 \cdots Q_{n-1}.$$

Matrix multiplication is used in the previous formula. For example, if $E = \{0, 1, \ldots, k\}$,

$$\begin{aligned}
&\alpha_0 Q_0 \\
=&(\alpha_0(0), \alpha_0(1), \ldots, \alpha_0(k)) \begin{pmatrix} Q_0(0,0) & Q_0(0,1) & \cdots & Q_0(0,k) \\ Q_0(1,0) & Q_0(1,1) & \cdots & Q_0(1,k) \\ \cdots & \cdots & \cdots & \cdots \\ Q_0(k,0) & Q_0(k,1) & \cdots & Q_0(k,k) \end{pmatrix} \\
=&\left(\sum_{i=0}^{k} \alpha_0(i) Q_0(i,0), \sum_{i=0}^{k} \alpha_0(i) Q_0(i,1), \ldots, \sum_{i=0}^{k} \alpha_n(i) Q_0(i,k)) \right) \\
=&\alpha_1,
\end{aligned}$$

$$
\begin{aligned}
&\alpha_0 Q_0 Q_1 = \alpha_1 Q_1 \\
=&\left(\sum_{i=0}^{k} \alpha_0(i) Q_0(i,0), \dots, \sum_{i=0}^{k} \alpha_0(i) Q_0(i,k))\right) \\
&\times \begin{pmatrix} Q_1(0,0) & \cdots & Q_1(0,k) \\ Q_1(1,0) & \cdots & Q_1(1,k) \\ \cdots & \cdots & \cdots \\ Q_1(k,0) & \cdots & Q_1(k,k) \end{pmatrix} \\
=&\left(\sum_{i=0}^{k} \sum_{j=0}^{k} \alpha_n(i) Q_0(i,j) Q_1(j,0), \dots, \sum_{i=0}^{k} \alpha_n(i) Q_0(i,j) Q_1(j,k)\right) \\
=&\alpha_2.
\end{aligned}
$$

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

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(i) $\mathbb{P}\{X_0 = 1\}$.

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Solution: (i) $\mathbb{P}\{X_0 = 1\} = 0.4$.

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(ii) $\mathbb{P}\{X_1 = 1\}$.

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(ii) $\mathbb{P}\{X_1 = 1\}$.

Solution: (ii) We have that

$$\mathbb{P}\{X_1 = 1\} = \sum_{j=0}^{2} \alpha_0(j) Q_1(j, 1) = (0.3)(0.3) + (0.4)(0.5) + (0.3)(0.1),$$

$= 0.32$, which is the second entry in the vector

$$(0.3, 0.4, 0.3) \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} = (0.21, 0.32, 0.47).$$

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Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(iii) $\mathbb{P}\{X_2 = 1\}$.

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Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

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(iii) $\mathbb{P}\{X_2 = 1\}$.

Solution: (iii) We have that

$$(0.3, 0.4, 0.3) \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

$$=(0.21, 0.32, 0.47) \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

$$=(0.258, 0.416, 0.326), \text{ and } \mathbb{P}\{X_2 = 1\} = 0.416.$$

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(iv) $\mathbb{P}\{X_0 = 1, X_1 = 2\}$.

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(iv) $\mathbb{P}\{X_0 = 1, X_1 = 2\}$.

Solution: (iv) $1 \mapsto 2$,

$$\mathbb{P}\{X_0 = 1, X_1 = 2\} = \alpha_0(1) Q_0(1, 2) = (0.4)(0.2) = 0.08.$$

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(v) $\mathbb{P}\{X_0 = 1, X_1 = 0\}$.

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(v) $\mathbb{P}\{X_0 = 1, X_1 = 0\}$.

Solution: (v) $1 \mapsto 0$,

$$\mathbb{P}\{X_0 = 1, X_1 = 0\} = \alpha_0(1)Q_0(1, 0) = (0.4)(0.3) = 0.12.$$

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(vi) $\mathbb{P}\{X_0 = 1, X_1 = 2, X_2 = 2\}$.

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(vi) $\mathbb{P}\{X_0 = 1, X_1 = 2, X_2 = 2\}$.

Solution: (vi)

$$\mathbb{P}\{X_0 = 1, X_1 = 2, X_2 = 2\} = \alpha_0(1) Q_0(1, 2) Q_1(2, 2)$$
$$=(0.4)(0.2)(0.2) = 0.016.$$

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(vii) $\mathbb{P}\{X_0 = 2, X_1 = 1, X_2 = 0\}$.

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(vii) $\mathbb{P}\{X_0 = 2, X_1 = 1, X_2 = 0\}$.

Solution: (vii) $2 \mapsto 1 \mapsto 0$,

$$\mathbb{P}\{X_0 = 2, X_1 = 1, X_2 = 0\} = \alpha_0(2) Q_0(2, 1) Q_1(1, 0)$$
$$= (0.3)(0.1)(0.3) = 0.009.$$

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(viii) $\mathbb{P}\{X_0 = 1, X_2 = 2\}$.

Example 4

Consider a Markov chain with $E = \{0, 1, 2\}$, $\alpha_0 = (0.3, 0.4, 0.3)$,

$$Q_0 = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

(viii) $\mathbb{P}\{X_0 = 1, X_2 = 2\}$.

Solution: (viii)

$$\mathbb{P}\{X_0 = 1, X_2 = 2\} = \sum_{j=0}^{2} \mathbb{P}\{X_0 = 1, X_1 = j, X_3 = 2\}$$

$$=(0.4)(0.3)(0.8) + (0.4)(0.5)(0.2) + (0.4)(0.2)(0.2) = 0.152.$$

Suppose that the annual effective rate of interest is i. Consider the cashflow

payments	C_1	C_2	$\cdots$	C_n
Time (in years)	t_1	t_2	$\cdots$	t_n

The **present value** of the former cashflow at time t is

$$\sum_{j=1}^{n} C_j(1+i)^{t-t_j}.$$

If the time at which the present value is omitted, we assume that the time is time zero. The present value of the former cashflow is

$$\sum_{j=1}^{n} C_j(1+i)^{-t_j}.$$

The annual interest factor is $1+i$.
The annual discount factor is $\nu = (1+i)^{-1}$.
The annual effective rate of discount is $d = 1-\nu = \frac{i}{1+i}$.

Suppose that the payments in a cashflow happen with certain probability. We have

Probability that a payment is made	p_1	p_2	$\cdots$	p_n
payments	C_1	C_2	$\cdots$	C_n
Time (in years)	t_1	t_2	$\cdots$	t_n

Then, the **actuarial present value** of the former cashflow is

$$\sum_{j=1}^{n} C_j p_j (1+i)^{-t_j}.$$

Example 5

An actuary models the life status of an individual with lung cancer using a non–homogenous Markov chain model with states: State 1: life; and State 2: dead. The transition probability matrices are

$$Q_0 = \begin{pmatrix} 0.6 & 0.4 \\ 0 & 1 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.4 & 0.6 \\ 0 & 1 \end{pmatrix}, Q_2 = \begin{pmatrix} 0.2 & 0.8 \\ 0 & 1 \end{pmatrix},$$
$$Q_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Suppose that changes in state occur at the end of the year. A death benefit of 100000 is paid at the end of the year of death. The annual effective rate of interest is 5%. The insured is alive at the beginning of year zero. Calculate the actuarial present value of this life insurance.

Solution: Since at the beginning the individual is alive, $\alpha_0 = (1, 0)$. We have that

$$\alpha_0 = \alpha_0 Q_0 = (1, 0) \begin{pmatrix} 0.6 & 0.4 \\ 0 & 1 \end{pmatrix} = (0.6, 0.4),$$

$$\alpha_1 = \alpha_0 Q_0 Q_1 = (0.6, 0.4) \begin{pmatrix} 0.4 & 0.6 \\ 0 & 1 \end{pmatrix} = (0.24, 0.76),$$

$$\alpha_2 = \alpha_0 Q_0 Q_1 Q_2 = (0.24, 0.76) \begin{pmatrix} 0.2 & 0.8 \\ 0 & 1 \end{pmatrix} = (0.048, 0.952),$$

$$\alpha_3 = \alpha_0 Q_0 Q_1 Q_2 Q_3 = (0.048, 0.952) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = (0, 1).$$

The probability that an individual dies in the n–th year is $\mathbb{P}\{X_n = 2\} - \mathbb{P}\{X_{n-1} = 2\}$.

Hence,

dead happens at the end of year	1	2	3	4
Probab.	0.4 0.4	$0.76 - 0.4$ $= 0.36$	$0.952 - 0.76$ $= 0.192$	$1 - 0.952$ $= 0.048$

The actuarial present value of this life insurance is

$$\begin{aligned} & 100000(0.4)(1.05)^{-1} + 100000(0.36)(1.05)^{-2} \\ & + 100000(0.192)(1.05)^{-3} + 100000(0.048)(1.05)^{-4} \\ = & 91282.95309. \end{aligned}$$

Theorem 6

(i)

$$P\{X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n | X_0 = i_0\}$$
$$= Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n).$$

(ii)

$$Q_0^{(n)}(i_0, i_n) = \sum_{i_1, \dots, i_{n-1} \in E} Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n).$$

Proof.

(i) Using Theorem 5, we have that

$$\begin{aligned}
&P\{X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n | X_0 = i_0\} \\
=&\frac{P\{X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n\}}{\mathbb{P}\{X_0 = i_0\}} \\
=&\frac{\alpha_0(i_0) Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n)}{\alpha_0(i_0)} \\
=&Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n).
\end{aligned}$$

(ii) Noticing that

$$\{X_n = i_n\} = \cup_{i_1,\dots,i_{n-1} \in E} \{X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i_n\},$$

where the union is over disjoint events, we get that

$$Q_0^{(n)}(i_0, i_n) = \sum_{i_1,\dots,i_{n-1} \in E} Q_0(i_0, i_1) Q_1(i_1, i_2) \cdots Q_{n-1}(i_{n-1}, i_n).$$

Using matrix notation, Theorem 6 (ii) states that

$$Q_0^{(n)} = Q_0 Q_1 \cdots Q_{n-1}. \tag{2}$$

This equation is one of the **Kolmogorov–Chapman equations** of a Markov chain.
For example, $Q_0^{(2)} = Q_0 Q_1$. Notice that

$$\begin{aligned}
&Q_0 Q_1 \\
=&\begin{pmatrix} Q_0(0,0) & \cdots & Q_0(0,k) \\ Q_0(1,0) & \cdots & Q_0(1,k) \\ \cdots & \cdots & \cdots \\ Q_0(k,0) & \cdots & Q_0(k,k) \end{pmatrix} \begin{pmatrix} Q_1(0,0) & \cdots & Q_1(0,k) \\ Q_1(1,0) & \cdots & Q_0(1,k) \\ \cdots & \cdots & \cdots \\ Q_1(k,0) & \cdots & Q_1(k,k) \end{pmatrix} \\
=&\begin{pmatrix} \sum_{i=0}^k Q_0(0,i) Q_1(i,0) & \dots & \sum_{i=0}^k Q_0(0,i) Q_0(i,k)) \\ \cdots & \cdots & \cdots \\ \sum_{i=0}^k Q_0(k,i)(i) Q_1(i,0) & \dots & \sum_{i=0}^k \alpha_n(i) Q_0(k,k)) \end{pmatrix} \\
=&Q_0^{(2)}.
\end{aligned}$$

Example 6

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_0 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Suppose that $X_0 = 1$.

Example 6

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_0 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Suppose that $X_0 = 1$.

(i) Find the probability that at stage 2 the chain is in state 2.

Example 6

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_0 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Suppose that $X_0 = 1$.

(i) Find the probability that at stage 2 the chain is in state 2.

Solution: (i) The Markov chain can be at stage 2 in state 1, if any of the following transitions occur

$$1 \mapsto 1 \mapsto 2,\ 1 \mapsto 2 \mapsto 2.$$

The probabilities of the previous occurrences are

$$(0.5)(0.8) = 0.4 \text{ and } (0.5)(0.4) = 0.2.$$

The probability that at stage 2 the chain is in state 1 is $0.4 + 0.2 = 0.6$.

Example 6

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_0 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Suppose that $X_0 = 1$.

(i) Find the probability that at stage 2 the chain is in state 2.

Solution: (i) We need to find $Q_0^{(2)}(1, 2) = \mathbb{P}\{X_2 = 1 | X_0 = 1\}$, which is the element $(1, 2)$ of the matrix:

$$Q_0 Q_1 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.6 \\ 0.48 & 0.52 \end{pmatrix}$$

The answer is 0.6.

Example 6

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_0 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Suppose that $X_0 = 1$.

(ii) Find the probability that the first time the chain is in state 2 is stage 2.

Example 6

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_0 = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}, Q_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}.$$

Suppose that $X_0 = 1$.

(ii) Find the probability that the first time the chain is in state 2 is stage 2.

Solution: (ii) If the first time the chain is in state 1 is stage 2, then the Markov chain does $1 \mapsto 1 \mapsto 2$, which happens with probability

$$\mathbb{P}\{X_1 = 1, X_2 = 2 | X_0 = 1\} = Q_0(1,1)Q_1(1,2) = (0.5)(0.8) = 0.4$$

Theorem 7

(i)

$$
\begin{aligned}
&P\{X_n = i_n, X_{n+1} = i_{n+1}, \dots, X_n = i_{n+m}\} \\
=&\alpha_n(i_n) Q_n(i_n, i_{n+1}) Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m}).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&\alpha_{n+m}(i_{n+m}) \\
=&\sum_{i_n, i_{n+1}, \dots, i_{n+m-1} \in E} \alpha_n(i_n) Q_n(i_n, i_{n+1}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m}).
\end{aligned}
$$

Proof: (i) Using Theorem 5,

$$\begin{aligned}
& P\{X_n = i_n, X_{n+1} = i_{n+1}, \dots, X_n = i_{n+m}\} \\
= & \sum_{i_0, i_1, \dots, i_{n-1}} P\{X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_{n+1} = i_{n+1}, \dots, X_n = i_{n+m}\} \\
= & \sum_{i_0, i_1, \dots, i_{n-1}} \alpha_0(i_0) Q_0(i_0, i_1) \cdots Q_{n-1}(i_{n-1}, i_n) \\
& \times Q_n(i_n, i_{n+1}) Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m}) \\
= & \alpha_n(i_n) Q_n(i_n, i_{n+1}) Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m}).
\end{aligned}$$

(ii) follows noticing that

$$\begin{aligned}
& \{X_{n+m} = i_{n+m}\} \\
= & \cup_{i_n, i_{n+1}, \dots, i_{n+m-1} \in E} \{X_n = i_n, X_{n+1} = i_{n+1}, \dots, X_{n+m} = i_{n+m}\},
\end{aligned}$$

where the union is over disjoint events.

In matrix notation

$$\alpha_{n+m}(i_{n+m}) = \sum_{i_n, i_{n+1}, \ldots, i_{n+m-1} \in E} \alpha_n(i_n) Q_n(i_n, i_{n+1}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m}).$$

says that

$$\alpha_{n+m} = \alpha_n Q_n Q_{n+1} \cdots Q_{n+m-1}.$$

Example 7

Consider a Markov chain with $E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:

Example 7

Consider a Markov chain with $E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:

(i) $\mathbb{P}\{X_3 = 2, X_4 = 1\}$.

Example 7

Consider a Markov chain with $E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(i) $\mathbb{P}\{X_3 = 2, X_4 = 1\}$.
Solution: (i)

$$\mathbb{P}\{X_3 = 2, X_4 = 1\} = \alpha_3(2)Q_3(2,1) = (0.8)(0.3) = 0.24.$$

Example 7

Consider a Markov chain with $E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(ii) $\mathbb{P}\{X_3 = 1, X_4 = 1, X_5 = 2\}$.

Example 7

Consider a Markov chain with $E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(ii) $\mathbb{P}\{X_3 = 1, X_4 = 1, X_5 = 2\}$.
Solution: (ii)

$$\mathbb{P}\{X_3 = 1, X_4 = 1, X_5 = 2\} = \alpha_3(1)Q_3(1,1)Q_4(1,2)$$
$$=(0.2)(0.6)(0.8) = 0.096.$$

Example 7

Consider a Markov chain with $E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(iii) $\mathbb{P}\{X_5 = 2\}$.

Example 7

Consider a Markov chain with $E = \{1, 2\}$, $\alpha_3 = (0.2, 0.8)$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(iii) $\mathbb{P}\{X_5 = 2\}$.
Solution: (iii) We have that

$$\alpha_5 = \alpha_3 Q_3 Q_4 = (0.2, 0.8) \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}$$
$$=(0.36, 0.64) \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix} = (0.52, 0.48).$$

Hence, $\mathbb{P}\{X_5 = 2\} = \alpha_5(2) = 0.48$.

Next theorem shows that for a Markov chain, given the present, the future is independent of the past, where future means events involving $X_{n+1}, \dots, X_{n+m}$, present means events involving X_n, and past means events involving $X_0, \dots, X_{n-1}$.

Theorem 8

Let $\{X_n : n = 0, 1, 2, \dots\}$ be Markov chain such that for each $i_0, i_1, \dots, i_n, j_1, \dots, j_m \in E$,

$$\begin{aligned}
&\mathbb{P}\{X_{n+1} = j_1, X_{n+2} = j_2, \dots, X_{n+m} = j_m | X_0 = i_0, \dots, X_n = i_n\} \\
=&Q_n(i_n, i_{n+1}) Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m}) \\
=&\mathbb{P}\{X_{n+1} = j_1, X_{n+2} = j_2, \dots, X_{n+m} = j_m | X_n = i_n\}.
\end{aligned}$$

Proof: By Lemma 1 and the definition of Markov chain, we have that

$$\begin{aligned}
&\mathbb{P}\{X_{n+1} = j_1, X_{n+2} = j_2, \dots, X_{n+m} = j_m | X_0 = i_0, \dots, X_n = i_n\} \\
=&\mathbb{P}\{X_{n+1} = j_1 | X_0 = i_0, X_1 = i_1, \dots, X_n = i_n\} \\
&\times \mathbb{P}\{X_{n+2} = j_2 | X_0 = i_0, X_1 = i_1, \dots, X_n = i_n, X_{n+1} = j_1\} \\
&\times \cdots \\
&\times \mathbb{P}\{X_{n+m} = j_m | X_0 = i_0, X_1 = i_1, \dots, X_{n+m-1} = j_{m-1}\} \\
=&\mathbb{P}\{X_{n+1} = j_1 | X_n = i_n\} \mathbb{P}\{X_{n+2} = j_2 | X_{n+1} = j_1\} \cdots \\
&\times \mathbb{P}\{X_{n+m} = j_m | X_{n+m-1} = j_{m-1}\} \\
=&Q_n(i_n, i_{n+1}) Q_{n+1}(i_{n+1}, i_{n+2}) \cdots Q_{n+m-1}(i_{n+m-1}, i_{n+m}).
\end{aligned}$$

By Theorem 7,

$$\begin{aligned}
&\mathbb{P}\{X_{n+1}=j_1, X_{n+2}=j_2,\dots,X_{n+m}=j_m|X_n=i_n\}\\
=&\frac{\mathbb{P}\{X_n=i_n, X_{n+1}=j_1, X_{n+2}=j_2,\dots,X_{n+m}=j_m\}}{\mathbb{P}\{X_n=i_n\}}\\
=&\frac{\alpha_n(i_n)Q_n(i_n,i_{n+1})Q_{n+1}(i_{n+1},i_{n+2})\cdots Q_{n+m-1}(i_{n+m-1},i_{n+m})}{\alpha_n(i_n)}\\
=&Q_n(i_n,i_{n+1})Q_{n+1}(i_{n+1},i_{n+2})\cdots Q_{n+m-1}(i_{n+m-1},i_{n+m}).
\end{aligned}$$

Previous theorem implies that:

Theorem 9

$$\mathbb{P}\{X_{n+m} = j_m | X_n = i_n\} = \sum_{j_1,\dots,j_{m-1}\in E} Q_n(i_n, j_1) Q_{n+1}(j_1, j_2) \cdots Q_{n+m-1}(j_{m-1}, j_m).$$

Proof:

$$\begin{aligned}
&\mathbb{P}\{X_{n+m} = j_m | X_n = i_n\} \\
&= \sum_{j_1,\dots,j_{m-1}\in E} \mathbb{P}\{X_{n+1} = j_1, X_{n+2} = j_2, \dots, X_{n+m} = j_m | X_n = i_n\} \\
&= \sum_{j_1,\dots,j_{m-1}\in E} Q_n(i_n, j_1) Q_{n+1}(j_1, j_2) \cdots Q_{n+m-1}(j_{m-1}, j_m).
\end{aligned}$$

Previous theorem in matrix notation says that:

Theorem 10

$$Q_n^{(m)} = Q_n Q_{n+1} \cdots Q_{n+m-1}.$$

Proof.

We have that

$$Q_n^{(m)}(i,j) = \mathbb{P}\{X_{n+m} = j | X_n = i\}$$
$$= \sum_{j_1,\dots,j_{m-1}\in E} Q_n(i,j_1) Q_{n+1}(j_1,j_2) \cdots Q_{n+m-1}(j_{m-1},j).$$

So, $Q_n^{(m)} = Q_n Q_{n+1} \cdots Q_{n+m-1}$. □

Equation

$$Q_n^{(m)} = Q_n Q_{n+1} \cdots Q_{n+m-1}$$

is one of the Kolmogorov–Chapman equations.

Example 8

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:

Example 8

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(i) $\mathbb{P}\{X_4 = 2 | X_3 = 1\}$.

Example 8

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(i) $\mathbb{P}\{X_4 = 2 | X_3 = 1\}$.
Solution: (i) $\mathbb{P}\{X_4 = 2 | X_3 = 1\} = Q_3(1, 2) = 0.4$.

Example 8

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(ii) $\mathbb{P}\{X_4 = 2, X_5 = 1 | X_3 = 1\}$.

Example 8

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(ii) $\mathbb{P}\{X_4 = 2, X_5 = 1 | X_3 = 1\}$.
Solution: (ii)

$$\mathbb{P}\{X_4 = 2, X_5 = 1 | X_3 = 1\} = Q_3(1, 2) Q_4(2, 1) = (0.4)(0.7) = 0.28.$$

Example 8

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:

(iii) $\mathbb{P}\{X_5 = 1 | X_3 = 1\}$.

Example 8

Consider a Markov chain with $E = \{1, 2\}$*,*

$$Q_3 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}, Q_4 = \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix}.$$

Find:
(iii) $\mathbb{P}\{X_5 = 1 | X_3 = 1\}$.
Solution: (iii) We have that

$$Q_3^{(2)} = Q_3 Q_4 = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 \\ 0.7 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.40 & 0.60 \\ 0.55 & 0.45 \end{pmatrix}$$

and

$$\mathbb{P}\{X_5 = 1 | X_3 = 1\} = Q_3^{(2)}(1, 1) = 0.4.$$

For a Markov chain, given the present, the future is independent of the past, where present means X_n, past means events involving $X_0, \dots, X_{n-1}$ and future means events involving $X_{n+1}, \dots, X_{n+m}$.

Theorem 11

Let $\{X_n : n = 0, 1, 2, \dots\}$ be Markov chain. Then, for each $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^m$

$$\begin{aligned}&\mathbb{P}\{(X_{n+1}, \dots, X_{n+m}) \in B | (X_0, \dots, X_{n-1}) \in A, X_n = i_n\}\\=&\mathbb{P}\{(X_{n+1}, \dots, X_{n+m}) \in B | X_n = i_n\}\end{aligned}$$

The proof of the previous theorem is in Arcones' manual.

Example 9

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_7 = \begin{pmatrix} 0.4 & 0.6 \\ 0.1 & 0.8 \end{pmatrix}, Q_8 = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix}.$$

Find $\mathbb{P}\{X_9 = 1 | X_5 = 2, X_7 = 1\}$.

Example 9

Consider a Markov chain with $E = \{1, 2\}$,

$$Q_7 = \begin{pmatrix} 0.4 & 0.6 \\ 0.1 & 0.8 \end{pmatrix}, Q_8 = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix}.$$

Find $\mathbb{P}\{X_9 = 1 | X_5 = 2, X_7 = 1\}$.

Solution: We have that

$$Q_7^{(2)} = Q_7 Q_8 = \begin{pmatrix} 0.4 & 0.6 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.48 & 0.52 \\ 0.51 & 0.39 \end{pmatrix}$$

and

$$\mathbb{P}\{X_9 = 1 | X_5 = 2, X_7 = 1\} = \mathbb{P}\{X_9 = 1 | X_7 = 1\} = Q_7^{(2)}(1, 1) = 0.48.$$

A Markov chain satisfying that $P(X_{n+1} = j | X_n = i)$ is independent of n is called an **homogeneous Markov chain**. We define $P(i,j) = \mathbb{P}\{X_{n+1} = j | X_n = i\}$. $P(i,j)$ is called the one-step transition probability from state i into state j. Notice that $P(i,j)$ does not depend on n.

We denote $P = (P(i,j))_{i,j \in E}$ the matrix consisting of the one-step transition probabilities. The matrix P must satisfy that

(i) for each $i, j \in E$, $P(i,j) \geq 0$.

(ii) for each $i \in E$, $\sum_{j \in E} P(i,j) = 1$.

We define $P^{(n)}(i,j) = \mathbb{P}\{X_{k+n} = j | X_k = i\}$. $P(i,j)$ is called the n–step transition probability from state i into state j. Notice that $P(i,j)$ does not depend on k. We denote $P^{(n)} = (P^{(n)}(i,j))_{i,j\in E}$ the matrix consisting of the n–step transition probabilities. We have that $P^{(1)} = P$. The matrix $P^{(n)}$ must satisfy that:
(i) for each $i,j \in E$, $P^{(n)}(i,j) \geq 0$.
(ii) for each $i \in E$, $\sum_{j\in E} P^{(n)}(i,j) = 1$.

Theorem 12

For an homogeneous Markov chain, we have

(i)

$$\begin{aligned}&\mathbb{P}\{X_n = i_n, X_{n+1} = i_{n+1}, \dots, X_{n+m-1} = i_{n+m-1}, X_{n+m} = i_{n+m}\}\\ =&\alpha_n(i_n)P(i_n, i_{n+1})P(i_{n+1}, i_{n+2}) \cdots P(i_{n+m-1}, i_{n+m}).\end{aligned}$$

(ii)

$$\begin{aligned}&\mathbb{P}\{X_{n+1} = i_{n+1}, \dots, X_{n+m-1} = i_{n+m-1}, X_{n+m} = i_{n+m} | X_n = i_n\}\\ =&P(i_n, i_{n+1})P(i_{n+1}, i_{n+2}) \cdots P(i_{n+m-1}, i_{n+m}).\end{aligned}$$

(iii)

$$P^{(n)} = P^n$$

(iv)

$$\alpha_{m+n} = \alpha_m P^n$$

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, transition matrix

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. Find:

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*

(i) $P^{(2)}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*

(i) $P^{(2)}$.

Solution: (i)

$$P^{(2)} = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.4444444 & 0.2222222 & 0.3333333 \\ 0.2916667 & 0.4583333 & 0.2500000 \\ 0.3333333 & 0.2916667 & 0.3750000 \end{pmatrix}$$

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*
(ii) $P^{(3)}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$*, transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$*. Find:*

(ii) $P^{(3)}$.

Solution: (ii)

$$P^{(3)} = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 0.4444444 & 0.2222222 & 0.3333333 \\ 0.2916667 & 0.4583333 & 0.2500000 \\ 0.3333333 & 0.2916667 & 0.3750000 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3425926 & 0.3796296 & 0.2777778 \\ 0.3888889 & 0.2569444 & 0.3541667 \\ 0.3506944 & 0.3159722 & 0.3333333 \end{pmatrix}$$

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$*, transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$*. Find:*

(iii) $\mathbb{P}\{X_2 = 2\}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*

(iii) $\mathbb{P}\{X_2 = 2\}$.

Solution: (iii)

$$\alpha_0 P^{(2)} = (1/2, 1/3, 1/6) \begin{pmatrix} 0.4444444 & 0.2222222 & 0.3333333 \\ 0.2916667 & 0.4583333 & 0.2500000 \\ 0.3333333 & 0.2916667 & 0.3750000 \end{pmatrix}$$

$=(0.375, 0.3125, 0.3125)$.

So, $\mathbb{P}\{X_2 = 2\} = 0.3125$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$*, transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$*. Find:*

(iv) $\mathbb{P}\{X_0 = 1, X_3 = 3\}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*
(iv) $\mathbb{P}\{X_0 = 1, X_3 = 3\}$.
Solution: (iv)

$$\mathbb{P}\{X_0 = 1, X_3 = 3\} = \mathbb{P}\{X_0 = 1\}\mathbb{P}\{X_3 = 3 | X_0 = 1\}$$
$$= \alpha_0(1)P^{(3)}(1, 3) = (0.5)(0.2777778) = 0.1388889.$$

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*

(v) $\mathbb{P}\{X_1 = 2, X_2 = 3, X_3 = 1 | X_0 = 1\}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$*, transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$*. Find:*

(v) $\mathbb{P}\{X_1 = 2, X_2 = 3, X_3 = 1 | X_0 = 1\}$.

Solution: (v)

$$\begin{aligned} &\mathbb{P}\{X_1 = 2, X_2 = 3, X_3 = 1 | X_0 = 1\} \\ =&P(1, 2)P(2, 3)P(3, 1) = (2/3)(1/2)(1/4) = 1/12. \end{aligned}$$

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$*, transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*
(vi) $\mathbb{P}\{X_2 = 3 | X_1 = 3\}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*
(vi) $\mathbb{P}\{X_2 = 3 | X_1 = 3\}$.
Solution: (vi)

$$\mathbb{P}\{X_2 = 3 | X_1 = 3\} = P(3, 3) = 1/2.$$

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*

(vii) $\mathbb{P}\{X_{12} = 1 | X_5 = 3, X_{10} = 1\}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$*, transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$*. Find:*

(vii) $\mathbb{P}\{X_{12} = 1 | X_5 = 3, X_{10} = 1\}$.

Solution: (vii)

$$\mathbb{P}\{X_{12} = 1 | X_5 = 3, X_{10} = 1\} = \mathbb{P}\{X(12) = 1 | X(10) = 1\}$$
$$= P^{(2)}(1, 1) = 0.444444.$$

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$*, transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$*. Find:*

(viii) $\mathbb{P}\{X_3 = 3, X_5 = 1 | X_0 = 1\}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, transition matrix

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. Find:
(viii) $\mathbb{P}\{X_3 = 3, X_5 = 1 | X_0 = 1\}$.
Solution: (viii)

$$\mathbb{P}\{X_3 = 3, X_5 = 1 | X_0 = 1\} = \mathbb{P}\{X_3 = 3 | X_0 = 1\}\mathbb{P}\{X_5 = 1 | X_3 = 3\}$$
$$= P^{(3)}(1,3)P^{(2)}(3,1) = (0.277778)(0.3333) = 0.09258341.$$

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, transition matrix

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. Find:

(ix) $\mathbb{P}\{X_3 = 3 | X_0 = 1\}$.

Example 10

Suppose that an homogeneous Markov chain has state space $E = \{1, 2, 3\}$, *transition matrix*

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix},$$

and initial distribution $\alpha_0 = (1/2, 1/3, 1/6)$. *Find:*

(ix) $\mathbb{P}\{X_3 = 3 | X_0 = 1\}$.

Solution: (ix)

$$\mathbb{P}\{X_3 = 3 | X_0 = 1\} = P^{(3)}(1, 3) = 0.277778.$$

