Manual for SOA Exam MLC.

Chapter 10. Markov chains.
Section 10.3. Random walk.

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Random walk

Let $\{\epsilon_j\}_{j=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with $\mathbb{P}\{\epsilon_j = 1\} = p$ and $\mathbb{P}\{\epsilon_j = -1\} = q = 1 - p$, where $0 < p < 1$. Given an integer X_0, let $X_n = X_0 + \sum_{j=1}^{n} \epsilon_j$ for $n \geq 1$. The stochastic process $\{X_n : n \geq 0\}$ is called a **random walk**. Usually $X_0 = 0$.

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A random walk is a homogeneous Markov chain with states $E = \{0, \pm 1, \pm 2, \dots\}$ and one–step transition probabilities, $P(i, i+1) = p$ and $P(i, i-1) = 1 - p$.

Imagine a drunkard coming back home at night. We assume that the drunkard goes in a straight line, but, he does not know which direction to take. After giving one step in one direction, the drunkard ponders which way to take home and decides randomly which direction to take. The drunkard's path is a random walk.

http://xanadu.math.utah.edu/java/brownianmotion/4/

http://webphysics.davidson.edu/WebTalks/clark/onedimensionalwalk.htm

Let $s_j = \frac{\epsilon_j + 1}{2}$. Since $\mathbb{P}\{\epsilon_j = 1\} = p$ and $\mathbb{P}\{\epsilon_j = -1\} = 1 - p$, we have that $\mathbb{P}\{s_j = 1\} = p$ and $\mathbb{P}\{s_j = 0\} = 1 - p$.
Since $s_1, \dots, s_n$ are independent r.v.'s, $S_n = \sum_{j=1}^n s_j$ has a binomial distribution with parameters n and p.

Theorem 1

(i) For each integer k with $0 \le k \le n$,

$$\mathbb{P}\{S_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}.$$

(ii) $E[S_n] = np$ and $\mathrm{Var}(S_n) = np(1-p)$.

Theorem 2

Suppose that $X_0 = 0$.
(i) For $-n \le j \le n$, with $\frac{n+j}{2}$ integer,

$$\mathbb{P}\{X_n = j\} = \mathbb{P}\left\{S_n = \frac{n+j}{2}\right\} = \binom{n}{\frac{n+j}{2}} p^{\frac{n+j}{2}} (1-p)^{\frac{n-j}{2}}.$$

(ii) $E[X_n] = E[2S_n - n] = 2E[S_n] - n = n(2p-1)$.
(iii) $\mathrm{Var}(X_n) = \mathrm{Var}(2S_n - n) = 4np(1-p)$.

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Proof: Since $s_j = \frac{\epsilon_j + 1}{2}$, $X_n = \sum_{j=1}^n \epsilon_j$ and $S_n = \sum_{j=1}^n s_j$, we have that $S_n = \frac{X_n + n}{2}$ and $X_n = 2S_n - n$.

Example 1

Suppose that $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability* 0.55 *of a step to the right, find:*
(i) $\mathbb{P}\{X_4 = -2\}$.
(ii) $E[X_4]$.
(iii) $\mathrm{Var}(X_4)$.

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Solution: (i)

$$\mathbb{P}\{X_4 = -2\} = \mathbb{P}\left\{\mathrm{Binom}(4, p) = \frac{4 + (-2)}{2}\right\}$$

$$= \mathbb{P}\{\mathrm{Binom}(4, p) = 1\} = \binom{4}{1}(0.55)^1(0.45)^3 = 0.200475.$$

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(ii) $E[X_4] = (4)((2)(0.55) - 1) = 0.4.$
(iii) $\mathrm{Var}(X_4) = (4)(4)(0.55)(0.45) = 3.96.$

Theorem 3

For each $1 \leq m \leq n$*,*
(i) S_m *and* $S_n - S_m$ *are independent.*
(ii) $S_n - S_m$ *has the distribution of* S_{n-m}*.*
(iii) $\mathrm{Cov}(S_m, S_n) = mp(1-p)$*.*

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(iii) $\mathrm{Cov}(S_m, S_n) = mp(1-p)$.

Proof.

(i) $S_m = \sum_{j=1}^{m} s_j$ and $S_n - S_m = \sum_{j=m+1}^{n} s_j$ depend on different on different s's, S_m and $S_n - S_m$ are independent.
(ii) Since $S_n - S_m = \sum_{j=m+1}^{n} s_j$ is the sum of $n-m$ s's, $S_n - S_m$ has the distribution of S_{n-m}.
(iii) Since S_m and $S_n - S_m$ are independent, $\mathrm{Cov}(S_m, S_n - S_m) = 0$. Hence,

$$\begin{aligned} &\mathrm{Cov}(S_m, S_n) = \mathrm{Cov}(S_m, S_m) + \mathrm{Cov}(S_m, S_n - S_m) \\ =&\mathrm{Cov}(S_m, S_m) = \mathrm{Var}(S_m) = mp(1-p). \end{aligned}$$

□

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(iii) $\mathrm{Cov}(S_m, S_n) = mp(1-p)$*.*

Using that $X_n - X_0 = 2S_n - n$, we get that

Theorem 4

For each $1 \leq m \leq n$*,*
(i) X_m *and* $X_n - X_m$ *are independent.*
(ii) $X_n - X_m$ *has the distribution of* $X_{n-m} - X_0$*.*
(iii) $\mathrm{Cov}(X_m, X_n) = 4mp(1-p)$*.*

Example 2

Suppose that $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability* p *of a step to the right, find:*

(i) $\mathbb{P}\{X_3 = -1, X_6 = 2\}$.

(ii) $\mathbb{P}\{X_5 = 1, X_{10} = 4, X_{16} = 2\}$.

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Solution:

(i)

$$\begin{aligned}
&\mathbb{P}\{X_3 = -1, X_6 = 2\} = \mathbb{P}\{X_3 = -1, X_6 - X_3 = 3\} \\
=&\mathbb{P}\{X_3 = -1\}\mathbb{P}\{X_6 - X_3 = 3\} = \mathbb{P}\{X_3 = -1\}\mathbb{P}\{X_3 = 3\} \\
=&\mathbb{P}\left\{\mathrm{Binom}(3, p) = \frac{3 + (-1)}{2}\right\}\mathbb{P}\left\{\mathrm{Binom}(3, p) = \frac{3 + 3}{2}\right\} \\
=&\mathbb{P}\{\mathrm{Binom}(3, p) = 1\}\,\mathbb{P}\{\mathrm{Binom}(3, p) = 3\} \\
=&\binom{3}{1}p^1q^2\binom{3}{3}p^3q^0 = 3p^4q^2.
\end{aligned}$$

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(i) $\mathbb{P}\{X_3 = -1, X_6 = 2\}$.
(ii) $\mathbb{P}\{X_5 = 1, X_{10} = 4, X_{16} = 2\}$.

Solution:
(ii)

$$\begin{aligned}
&\mathbb{P}\{X_5 = 1, X_{10} = 4, X_{16} = 2\} \\
=&\mathbb{P}\{X_5 = 1, X_{10} - X_5 = 3, X_{16} - X_{10} = -2\} \\
=&\mathbb{P}\{X_5 = 1\}\mathbb{P}\{X_5 = 3\}\mathbb{P}\{X_6 = -2\} \\
=&\mathbb{P}\{\mathrm{Binom}(5, p) = (5+1)/2\}\mathbb{P}\{\mathrm{Binom}(5, p) = (5+3)/2\} \\
&\mathbb{P}\{\mathrm{Binom}(6, p) = (6-2)/2\} \\
=&\binom{5}{3} p^3 q^2 \binom{5}{4} p^4 q^1 \binom{6}{2} p^2 q^4 = 750 p^9 q^7.
\end{aligned}$$

Example 3

Suppose that $\{X_n : n \geq 1\}$ *is a random walk with* $X_0 = 0$ *and probability p of a step to the right, find:*
(i) $\mathrm{Var}(-3 + 2X_4)$
(ii) $\mathrm{Var}(-2 + 3X_2 - 2X_5)$.
(iii) $\mathrm{Var}(3X_4 - 2X_5 + 4X_{10})$.

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(iii) $\mathrm{Var}(3X_4 - 2X_5 + 4X_{10})$.

Solution:

(i)

$$\mathrm{Var}(-3 + 2X_4) = (2)^2 \mathrm{Var}(X_4) = (2)^2 (4)(4)p(1 - p) = 64p(1 - p).$$

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(i) $\mathrm{Var}(-3 + 2X_4)$
(ii) $\mathrm{Var}(-2 + 3X_2 - 2X_5)$.
(iii) $\mathrm{Var}(3X_4 - 2X_5 + 4X_{10})$.

Solution:
(ii)

$$\begin{aligned}
&\mathrm{Var}(-2 + 3X_2 - 2X_5) = \mathrm{Var}(3X_2 - 2X_5) \\
=&\mathrm{Var}((3-2)X_2 - 2(X_5 - X_2)) \\
=&\mathrm{Var}(X_2 - 2(X_5 - X_2)) = \mathrm{Var}(X_2) + (-2)^2\mathrm{Var}(X_5 - X_2) \\
=&(4)(2)p(1-p) + (-2)^2(4)(3)p(1-p) = 56p(1-p).
\end{aligned}$$

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Suppose that $\{X_n : n \geq 1\}$ is a random walk with $X_0 = 0$ and probability p of a step to the right, find:
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(ii) $\mathrm{Var}(-2 + 3X_2 - 2X_5)$.
(iii) $\mathrm{Var}(3X_4 - 2X_5 + 4X_{10})$.

Solution:
(iii)

$$
\begin{aligned}
&\mathrm{Var}(3X_4 - 2X_5 + 4X_{10}) \\
=&\mathrm{Var}((3 - 2 + 4)X_4 + (-2 + 4)(X_5 - X_4) + 4(X_{10} - X_5)) \\
=&\mathrm{Var}(5X_4 + 2(X_5 - X_4) + 4(X_{10} - X_5)) \\
=&\mathrm{Var}(5X_4) + \mathrm{Var}(2(X_5 - X_4) + \mathrm{Var}(4(X_{10} - X_5)) \\
=&(5)^2\mathrm{Var}(X_4) + (2)^2\mathrm{Var}(X_5 - X_4) + (4)^2\mathrm{Var}(X_{10} - X_5) \\
=&(5)^2(4)(4)p(1 - p) + (2)^2(4)(1)p(1 - p) + (4)^2(4)(5)p(1 - p) \\
=&736p(1 - p)
\end{aligned}
$$

Gambler's ruin problem.

Imagine a game played by two players. Player A starts with \$$k$ and player B starts with \$$(N - k)$. They play successive games until one of them ruins. In every game, they bet \$. The probability that A wins is p and the probability that A losses is $1 - p$. Assume that the outcomes in different games are independent. Let X_n be player A's money after n games. After, one of the player ruins, no more wagers are done.

If $X_n = N$, then $X_m = N$, for each $m \geq n$. If $X_n = 0$, then $X_m = 0$, for each $m \geq n$. For $1 \leq k \leq N-1$,

$$\mathbb{P}\{X_{n+1} = k+1 | X_n = k\} = p, \ \mathbb{P}\{X_{n+1} = k-1 | X_n = k\} = 1-p.$$

$\{X_n\}_{n=1}^{\infty}$ is a homogeneous Markov chain with states $\{0, 1, \dots, N\}$ and one–step transition probability

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & 0 & \cdots & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}.$$

http://probability.ca/jeff/java/gambler.html

Let P_k be the probability that player A wins (player B gets broke). We have that

$$P_k = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} & \text{if } p \neq \frac{1}{2} \\ \frac{k}{N} & \text{if } p = \frac{1}{2} \end{cases}$$

P_k is the probability that a random walk with $X_0 = k$, reaches N before reaching 0. P_k is the probability that a random walk goes up $N - k$ before it goes down k. If $N \to \infty$,

$$P_k = \begin{cases} 1 - \left(\frac{q}{p}\right)^k & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p \leq \frac{1}{2} \end{cases}$$

Now, P_k is the probability that a random walk with $X_0 = k$, where $k > 0$, never reaches 0.

Example 4

Two gamblers, A and B make a series of \$1 wagers where B has 0.55 chance of winning and A has a 0.45 chance of winning on each wager. What is the probability that B wins \$10 before A wins \$5?

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Solution: Here, $p = 0.55$, $k = 5$, $N - k = 10$. So, the probability that B wins \$10 before A wins \$5 is

$$\frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N} = \frac{1 - \left(\frac{0.45}{0.55}\right)^5}{1 - \left(\frac{0.45}{0.55}\right)^{15}}.$$

