Manual for SOA Exam MLC.

Chapter 10. Poisson processes.

Section 10.5. Nonhomogenous Poisson processes

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Nonhomogenous Poisson processes

Definition 1

The counting process $\{N(t) : t \geq 0\}$ *is said to be a* **nonhomogenous Poisson process** *with* **intensity function** $\lambda(t)$, $t \geq 0$, *if*

(i) $N(0) = 0$.

(ii) For each $t > 0$, $N(t)$ *has a Poisson distribution with mean* $m(t) = \int_0^t \lambda(s)\, ds$.

(iii) For each $0 \leq t_1 < t_2 < \cdots < t_m$, $N(t_1), N(t_2) - N(t_1), \dots, N(t_m) - N(t_{m-1})$ *are independent r.v.'s.*

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- An nonhomogeneous Poisson process with $\lambda(t) = \lambda$, for each $t \geq 0$, is a regular Poisson process.
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- For each $0 \leq t_1 < t_2 < \cdots < t_m$ and each integers $k_1, \dots, k_m \geq 0$,

$$\begin{aligned} & \mathbb{P}\{N(t_1) = k_1, N(t_2) = k_2, \dots, N(t_m) = k_m\} \\ = \; & \mathbb{P}\{N(t_1) = k_1, N(t_2) - N(t_1) = k_1 - k_1, \dots, \\ & N(t_m) - N(t_{m-1}) = k_m - k_{m-1}\} \\ = \; & \frac{e^{-m(t_1)}(m(t_1))^{k_1}}{k_1!} \frac{e^{-(m(t_2)-m(t_1))}(m(t_2)-(m(t_1))^{k_2-k_1}}{(k_2-k_1)!} \cdots \\ & \frac{e^{-(m(t_m)-m(t_{m-1}))}(m(t_m)-m(t_{m-1}))^{k_m-k_{m-1}}}{(k_m-k_{m-1})!}, \end{aligned}$$

Example 1

For a nonhomogenous Poisson process the intensity function is given by

$$\lambda(t) = \begin{cases} 5 & \text{if } t \text{ is in } (1,2], (3,4], \dots \\ 3 & \text{if } t \text{ is in } (0,1], (2,3], \dots \end{cases}$$

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Find the probability that the number number of observed occurrences in the time period $(1.25, 3]$ *is more than two.*

Solution: $N(3) - N(1.25)$ has a Poisson distribution with mean

$$m(3) - m(1.25) = \int_{1.25}^{3} \lambda(t)\, dt = \int_{1.25}^{2} 5\, dt + \int_{2}^{3} 3\, dt = 6.75.$$

Hence,

$$\mathbb{P}\{N(3) - N(1.25) > 2\} = 1 - e^{-6.75}(1 + 6.75 + (6.75)^2/2)$$
$$= 0.9642515816.$$

Let S_n be the time of the n–th occurrence. S_n is an extended r.v. with values in $[0, \infty]$. Then,

$$\begin{aligned} & \mathbb{P}\{S_n > t\} = \mathbb{P}\{N(t) \le n-1\} = \sum_{j=0}^{n-1} \frac{e^{-m(t)}(m(t))^j}{j!} \\ = \; & \mathbb{P}\{\mathrm{Gamma}(n,1) \ge m(t)\}. \end{aligned}$$

If $\lim_{t\to\infty} m(t) = \infty$, then

$$\mathbb{P}\{S_n = \infty\} = \lim_{t\to\infty} \mathbb{P}\{S_n > t\} = \mathbb{P}\{\mathrm{Gamma}(n,1) \ge \lim_{t\to\infty} m(t)\} = 0$$

and S_n is a r.v. The density of S_n is

$$f_{S_n}(t) = e^{-m(t)} \frac{(m(t))^{n-1} m'(t)}{(n-1)!} = e^{-m(t)} \frac{(m(t))^{n-1} \lambda(t)}{(n-1)!}, t \ge 0.$$

If $\lim_{t\to\infty} m(t) < \infty$, then S_n is a mixed r.v. with

$$\mathbb{P}\{S_n = \infty\} = \mathbb{P}\{\mathrm{Gamma}(n,1) \ge \lim_{t\to\infty} m(t)\} > 0$$

and density of its continuous part $f_{S_n}(t) = \frac{e^{-m(t)}(m(t))^{n-1}\lambda(t)}{(n-1)!}$, $t \ge 0$.

Let $T_n = S_n - S_{n-1}$ be the n–th interarrival time. For a non–homogeneous Poisson process $\{T_n\}_{n=1}^{\infty}$ are not necessarily independent r.v.'s. For $0 \le s \le t$,

$$\begin{aligned} &\mathbb{P}\{T_{n+1} > t | S_n = s\} = \mathbb{P}\{S_{n+1} > s + t | S_n = t\} \\ =&\mathbb{P}\{N(s+t) = n+1 | S_n = t\} = \mathbb{P}\{N(s+t) - N(s) = 1 | S_n = t\} \\ =&\mathbb{P}\{N(s+t) - N(s) = 1\} = e^{-(m(s+t)-m(s))}. \end{aligned}$$

Notice $S_n = t$ depends on the non–homogeneous Poisson process until time t. Hence, $\{N(s+t) - N(s) = 1\}$ and $\{S_n = t\}$ are independent.

Example 2

For a non–homogenous Poisson process, the intensity function is given by

$$\lambda(t) = \begin{cases} t & \text{for } 0 < t \leq 4, \\ 4 & \text{for } 10 < t. \end{cases}$$

If $S_5 = 2$*, calculate the probability that* $S_6 > 5$*.*

Example 2

For a non–homogenous Poisson process, the intensity function is given by

$$\lambda(t) = \begin{cases} t & \text{for} \quad 0 < t \leq 4, \\ 4 & \text{for} \quad 10 < t. \end{cases}$$

If $S_5 = 2$*, calculate the probability that* $S_6 > 5$*.*

Solution: We have that

$$\mathbb{P}\{S_6 > 5 | S_5 = 2\} = \mathbb{P}\{N(5) = 5 | S_5 = 2\}$$
$$= \mathbb{P}\{N(5) - N(2) = 0 | S_5 = 2\} = P\{N(5) - N(2) = 0\}$$

and

$$m(5) - m(2) = \int_2^5 \lambda(t)\, dt = \int_2^4 t\, dt + \int_4^5 4\, dt = 10.$$

Hence, $\mathbb{P}\{S_6 > 5 | S_5 = 2\} = P\{N(5) - N(2) = 0\} = e^{-10}$.

