Manual for SOA Exam MLC.

Chapter 11. Poisson processes.

Section 11.1. Exponential and gamma distributions.

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Exponential distribution

Definition 1

The **gamma function** *is defined by*

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$\Gamma(\alpha) < \infty$, for $\alpha > 0$, because

$$\int_0^\infty t^{\alpha-1} e^{-t}\, dt = \int_0^1 t^{\alpha-1} e^{-t}\, dt + \int_1^\infty t^{\alpha-1} e^{-t}\, dt$$

$$\leq \int_0^1 t^{\alpha-1}\, dt + \int_1^\infty e^{-t}\, dt = \frac{t^\alpha}{\alpha}\Big|_0^1 + e^{-t}\Big|_1^\infty = \frac{1}{\alpha} + e^{-1}.$$

Theorem 1

The gamma function satisfies the following properties:
(i) For each $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
(ii) For each integer $n \geq 1$, $\Gamma(n) = (n - 1)!$.
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Proof:

(i) For each $\alpha > 1$, by an integration by parts

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt = \int_0^\infty t^{\alpha-1}d(-e^{-t})\,dt$$

$$=t^{\alpha-1}(-e^{-t})\Big|_0^\infty - \int_0^\infty (-e^{-t})d(t^{\alpha-1})$$

$$=\int_0^\infty (\alpha-1)t^{\alpha-2}e^{-t}\,dt = (\alpha-1)\Gamma(\alpha-1).$$

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Proof:
(ii) We use induction to prove that for each integer $n \geq 1$, $\Gamma(n) = (n-1)!$. We have that $\Gamma(1) = 1$. The case $n-1$ implies the case n, because if $\Gamma(n-1) = (n-2)!$, then $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)! = (n-1)!$.

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Proof:
(iii) By the change of variables $t = x^2/2$, $(dt = x\,dx)$,

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\,dt = \int_0^\infty \sqrt{\frac{2}{x^2}} e^{-\frac{x^2}{2}} x\,dx = \sqrt{2}\int_0^\infty e^{-\frac{x^2}{2}}\,dx$$
$$= \frac{\sqrt{2}}{2}\int_{-\infty}^\infty e^{-\frac{x^2}{2}}\,dx = \frac{\sqrt{2}}{2}\sqrt{2\pi} = \sqrt{\pi}$$

In particular, we have that

$$\int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1) = n!, n \geq 1. \tag{1}$$

By the change of variables $y = \frac{x}{\theta}$,

$$\int_0^\infty x^{\alpha-1} e^{-x/\theta}\,dx = \int_0^\infty y^{\alpha-1}\theta^{\alpha} e^{-y}\,dy = \theta^{\alpha}\Gamma(\alpha), \alpha, \theta > 0. \tag{2}$$

Example 1

Find:

(i) $\int_0^\infty x^3 e^{-x}\, dx$.

(ii) $\int_0^\infty x^{12} e^{-x}\, dx$.

(iii) $\int_0^\infty x^{23} e^{-2x}\, dx$.

(iv) $\int_0^\infty x^{24} e^{-x/3}\, dx$.

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(v) $\int_0^\infty x^{3.5} e^{-x}\,dx = \Gamma(3.5) = (2.5)\Gamma(2.5) = (2.5)(1.5)\Gamma(1.5) = (2.5)(1.5)(0.5)\sqrt{\pi}$.

Theorem 2

For each integer $n \geq 1$,

$$\int \frac{x^n}{n!} e^{-x}\, dx = -e^{-x} \sum_{j=0}^{n} \frac{x^j}{j!} + c.$$

Proof.

We proceed by induction. If $n = 1$, we have that

$$\int xe^{-x}\,dx = \int x d(-e^{-x}) = x(-e^{-x}) - \int (-e^{-x})\,dx$$
$$= x(-e^{-x}) - e^{-x} + c = -e^{-x}(1+x) + c.$$

Assume that the claim holds for the case $n-1$. Then,

$$\int \frac{x^n}{n!} e^{-x}\,dx = \int \frac{x^n}{n!} d(-e^{-x}) = \frac{x^n}{n!}(-e^{-x}) - \int (-e^{-x})\,d\left(\frac{x^n}{n!}\right)$$
$$= -\frac{x^n}{n!} e^{-x} + \int e^{-x} \frac{x^{n-1}}{(n-1)!}\,dx = -\frac{x^n}{n!} e^{-x} - \sum_{j=0}^{n-1} e^{-x} \frac{x^j}{j!} + c$$
$$= -\sum_{j=0}^{n} e^{-x} \frac{x^j}{j!} + c.$$

□

Definition 2

A r.v. X is said to have an **exponential distribution** *with parameter $\lambda > 0$, if the density of X is given by*

$$f(x) = \begin{cases} \frac{e^{-\frac{x}{\lambda}}}{\lambda} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We denote this by $X \sim \mathrm{Exponential}(\lambda)$.

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The above function f defines a legitime density because it is nonnegative and

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} \frac{e^{-\frac{t}{\lambda}}}{\lambda}\,dt = -e^{-\frac{t}{\lambda}} \Big|_0^{\infty} = 1.$$

Theorem 3

Let X be a r.v. with an exponential distribution with parameter $\lambda > 0$, then

$$E[X] = \lambda, \mathrm{Var}(X) = \lambda^2, E[X^k] = \lambda^k k!, M(t) = \frac{1}{1-\lambda t}, \text{ if } t < \lambda^{-1}.$$

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Proof: Using that

$$\int_0^\infty x^{\alpha-1} e^{-x/\theta}\, dx = \theta^\alpha \Gamma(\alpha),$$

we get that

$$E[X^k] = \int_0^\infty x^k \frac{e^{-\frac{x}{\lambda}}}{\lambda}\, dx = \frac{1}{\lambda}\Gamma(k+1)\lambda^{k+1} = k!\lambda^k$$

In particular, $E[X] = \lambda$ and $E[X^2] = 2\lambda^2$. Hence,

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 2\lambda^2 - \lambda^2 = \lambda^2.$$

We have that for $t < \lambda^{-1}$,

$$M(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{e^{-\frac{x}{\lambda}}}{\lambda} \, dx = \frac{1}{\lambda} \int_0^\infty e^{-x(\frac{1-\lambda t}{\lambda})} \, dx$$

$$= \frac{1}{\lambda} \frac{\lambda}{1-\lambda t} = \frac{1}{1-\lambda t}.$$

Example 2

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Solution: Let m be the median of the failure time. The failure time has density $\frac{1}{1000}e^{-\frac{x}{1000}}$. Hence,

$$\frac{1}{2} = \int_m^\infty \frac{1}{1000}e^{-\frac{x}{1000}}\,dx = -e^{-\frac{x}{1000}}\Big|_m^\infty = e^{-\frac{m}{1000}}$$

and $m = 1000\log(2) = 693.1471806$.

The cumulative distribution function of an exponential distribution with mean $\lambda > 0$ is

$$F(x) = \mathbb{P}\{X \leq x\} = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \frac{e^{-\frac{t}{\lambda}}}{\lambda}\,dt = 1 - e^{-\frac{x}{\lambda}},\ x \geq 0.$$

The exponential distribution satisfies that for each $s, t \geq 0$,

$$\mathbb{P}\{X > s + t | X > t\} = \mathbb{P}\{X > s\}.$$

This is property is called the **memoryless property of the exponential distribution**. Notice that

$$\mathbb{P}\{X > s+t | X > t\} = \frac{\mathbb{P}\{X > s + t\}}{\mathbb{P}\{X > t\}} = \frac{e^{-\frac{s+t}{\lambda}}}{e^{-\frac{t}{\lambda}}} = e^{-\frac{s}{\lambda}} = \mathbb{P}\{X > s\}.$$

Example 3

The life time T of a radioactive substance is an exponential distributed random variable with mean 1.5 *years.*
(i) What is the probability that the lifetime of a sample of this substance exceeds 2 *years.*
(ii) What is the probability that a sample of radioactive substance is present 14 years given that is duration exceeds 12 *years.*

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Solution: (i) T has density $f(t) = \frac{2}{3}e^{-\frac{2t}{3}}$, $t \geq 0$. So,

$$\mathbb{P}\{T \geq 2\} = \int_2^\infty \frac{2}{3}e^{-\frac{2t}{3}}\,dt = -e^{-\frac{2t}{3}}\Big|_2^\infty = e^{-\frac{4}{3}},$$

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(ii)

$$\mathbb{P}\{T \geq 14 | T \geq 12\} = \mathbb{P}\{T \geq 2\} = e^{-\frac{4}{3}}.$$

Definition 3

X has a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, if the density of X is

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We denote this by $X \sim \mathrm{Gamma}(\alpha, \beta)$.

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The above function f defines a legitime density because by (2)

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A gamma distribution with parameter $\alpha = 1$ is an exponential distribution.

Theorem 4

If X has a gamma distribution with parameters α and β, then

$$E[X] = \alpha\beta, \mathrm{Var}(X) = \alpha\beta^2, E[X^k] = \frac{\Gamma(\alpha + k)\beta^k}{\Gamma(\alpha)},$$

$$M(t) = \frac{1}{(1 - \beta t)^\alpha}, \text{ if } t < \frac{1}{\beta}.$$

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$$M(t) = \frac{1}{(1 - \beta t)^\alpha}, \text{ if } t < \frac{1}{\beta}.$$

Proof: Using that

$$\int_0^\infty x^{\alpha-1} e^{-x/\theta}\, dx = \theta^\alpha \Gamma(\alpha),$$

$$E[X^k] = \int_0^\infty x^k \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}\, dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{k+\alpha-1} e^{-\frac{x}{\beta}}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(k + \alpha)\beta^{k+\alpha} = \frac{\Gamma(\alpha + k)\beta^\alpha}{\Gamma(\alpha)}.$$

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Proof: Using that $E[X^k] = \frac{\Gamma(\alpha+k)\beta^\alpha}{\Gamma(\alpha)}$,

$$E[X] = \frac{\Gamma(\alpha + 1)\beta^1}{\Gamma(\alpha)} = \alpha\beta, E[X^2] = \frac{\Gamma(\alpha + 2)\beta^2}{\Gamma(\alpha)} = (\alpha + 1)\alpha\beta^2.$$

Hence,

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = (\alpha + 1)\alpha\beta^2 - (\alpha\beta)^2 = \alpha\beta^2.$$

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$$M(t) = \frac{1}{(1-\beta t)^\alpha}, \text{ if } t < \frac{1}{\beta}.$$

Proof: We have that for $t < \frac{1}{\beta}$,

$$M(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1-\beta t}{\beta}\right)}\, dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha)$$

$$= \frac{1}{(1-\beta t)^\alpha}.$$

Example 4

Find the mean and the variance of a random variable X with density

$$f(x) = \frac{x^3 e^{-\frac{x}{3}}}{486}, x \geq 0.$$

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$$f(x) = \frac{x^3 e^{-\frac{x}{3}}}{486}, x \geq 0.$$

Solution: We have that $\alpha = 4$, and $\beta = 3$. Hence,

$$E[X] = \alpha\beta = 12 \text{ and } \mathrm{Var}(X) = \alpha\beta^2 = 36.$$

Theorem 5

Let $X \sim \mathrm{Gamma}(\alpha_1, \beta)$ *and* $Y \sim \mathrm{Gamma}(\alpha_2, \beta)$. *Suppose that* X *and* Y *are independent. Then,* $X + Y \sim \mathrm{Gamma}(\alpha_1 + \alpha_2, \beta)$

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Proof.

The moment generating function of X and Y are $M_X(t) = \frac{1}{(1-\beta t)^{\alpha_1}}$ and $M_Y(t) = \frac{1}{(1-\beta t)^{\alpha_2}}$, respectively. Since X and Y are independent r.v.'s, the moment generating function of $X + Y$ is

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \frac{1}{(1-\beta t)^{\alpha_1}} \frac{1}{(1-\beta t)^{\alpha_2}}$$

$$= \frac{1}{(1-\beta t)^{\alpha_1+\alpha_2}},$$

which is the moment generating function of a gamma distribution with parameters $\alpha_1 + \alpha_2$ and β. So, $X + Y$ has a gamma distribution with parameters $\alpha_1 + \alpha_2$ and β. □

Notice that by induction, the previous theorem implies that if $X_1, \dots, X_n$ are independent r.v.'s and $X_i \sim \mathrm{Gamma}(\alpha_i, \beta)$, $1 \le i \le n$, then, $\sum_{i=1}^n X_i \sim \mathrm{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$.

Theorem 6

Let $X_1, \ldots, X_n$ *be independent identically distributed r.v.'s with exponential distribution with parameter* λ*. Then,* $\sum_{i=1}^{n} X_i$ *has a gamma distribution with parameters* n *and* λ*.*

Theorem 6

Let $X_1, \ldots, X_n$ be independent identically distributed r.v.'s with exponential distribution with parameter λ. Then, $\sum_{i=1}^{n} X_i$ has a gamma distribution with parameters n and λ.

Proof.

Every X_i has Gamma$(1, \lambda)$ distribution. By the previous theorem, $\sum_{i=1}^{n} X_i$ has a Gamma(n, λ) distribution. □

Example 5

Suppose that you arrive at a single–teller office of the Department of Motor Vehicles to find three customers waiting in line and one being served. If the services times are all exponential with rate 2 *minutes, calculate the probability that you will have to wait in line more than 10 minutes before being served.*

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Solution: By the memory–less property of the exponential distribution, the remaining serving time for the customer which is served is also exponential. Hence, your waiting time is $Y = \sum_{j=1}^{4} X_j$, where $\{X_j\}_{j=1}^{4}$ are independent identically distributed r.v.'s with an exponential distribution with mean 2. By Theorem 6, Y has a gamma distribution with parameters 4 and 2. Hence, the density of Y is

$$f_Y(y) = \frac{y^3 e^{-\frac{y}{2}}}{2^4(3!)} = \frac{y^3 e^{-\frac{y}{2}}}{96}.$$

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Solution: By the change of variable $\frac{y}{2} = z$,

$$\mathbb{P}\{Y \geq 10\} = \int_{10}^{\infty} \frac{y^3 e^{-\frac{y}{2}}}{96}\, dy = \int_{5}^{\infty} \frac{z^3 e^{-z}}{6}\, dz$$
$$= -\left(\frac{z^3}{6} + \frac{z^2}{2} + z + 1\right) e^{-z} \Big|_5^{\infty} = \left(\frac{5^3}{6} + \frac{5^2}{2} + 5 + 1\right) e^{-5}$$
$$= 0.2650259153.$$

Suppose that a system has n parts. The system functions works only if all n parts work. Let X_i is the lifetime of the i–th part of the system. Suppose that $X_1, \dots, X_n$ be independent r.v.'s and that X_i has an exponential distribution with mean θ_i. Let Y be the lifetime of the system. Then, $Y = \min(X_1, \dots, X_n)$. Then,

$$\mathbb{P}\{Y > t\} = \mathbb{P}\{\min(X_1, \dots, X_n) > t\} = \mathbb{P}\{\cap_{i=1}^{n}\{X_i > t\}\}$$
$$= \prod_{i=1}^{n} \mathbb{P}[X_i > t\} = \prod_{i=1}^{n} e^{-\frac{t}{\theta_i}} = e^{-t\sum_{i=1}^{n}\frac{1}{\theta_i}}.$$

So, Y has an exponential distribution with mean $\frac{1}{\sum_{i=1}^{n}\frac{1}{\theta_i}}$.

Example 6

A system consists of 4 components. The lifetime of the four component are independent random variables with an exponential distribution and respective means 2,3,4, 10. The system will work only if all four components work. Find the expected lifetime of the system.

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Solution: Let X_i, $1 \leq i \leq 4$, are the lifetime of the components. Let $T = \min_{1 \leq i \leq 4} X_i$ be the lifetime of the system. We have that

$$E[T] = \frac{1}{\sum_{i=1}^{n} \frac{1}{\theta_i}} = \frac{1}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{10}} = 0.8450704225.$$

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$$E[T] = \frac{1}{\sum_{i=1}^{n} \frac{1}{\theta_i}} = \frac{1}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{10}} = 0.8450704225.$$

Theorem 7

(i) Let X and Y be two independent r.v.'s such that X has an exponential distribution with mean θ_1 and Y has an exponential distribution with mean θ_2. Then,

$$\mathbb{P}\{X < Y\} = \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}}.$$

(ii) Let $X_1, \ldots, X_n$ be independent r.v.'s such that X_i has an exponential distribution with mean θ_i. Then,

$$\mathbb{P}\{X_i = \min_{1 \le j \le n} X_j\} = \frac{\frac{1}{\theta_i}}{\sum_{j=1}^{n} \frac{1}{\theta_j}}.$$

Proof: (i) The joint density of X and Y is

$$f_{X,Y}(x,y) = \frac{e^{-\frac{x}{\theta_1}-\frac{y}{\theta_2}}}{\theta_1\theta_2}, x, y > 0.$$

So,

$$\mathbb{P}\{X < Y\} = \int_0^\infty \int_x^\infty \frac{e^{-\frac{x}{\theta_1}-\frac{y}{\theta_2}}}{\theta_1\theta_2}\,dy\,dx = \int_0^\infty \frac{e^{-\frac{x}{\theta_1}-\frac{x}{\theta_2}}}{\theta_1}\,dx = \frac{\frac{1}{\theta_1}}{\frac{1}{\theta_1}+\frac{1}{\theta_2}}$$

Proof: (ii)

$$\mathbb{P}\{X_i = \min_{1\le j\le n} X_j\} = \mathbb{P}\{X_i < \min_{1\le j\le n, j\ne i} X_j\} = \frac{\frac{1}{\theta_i}}{\sum_{j=1}^{n} \frac{1}{\theta_j}},$$

because $\min_{1\le j\le n, j\ne i} X_j$ has an exponential distribution with mean $\left(\sum_{1\le j\le n, j\ne i}^{n} \frac{1}{\theta_j}\right)^{-1}$ and X_i and $\min_{1\le j\le n, j\ne i} X_j$ are independent.

Example 8

A factory has two electricity generators. The smaller of the two generators has expected duration before failure of 20 days. The other generator has an expected duration of 15 *days. The amount of time which each generator lasts before failing has an exponential distribution. The duration before failure of the two generators are independent r.v.'s.*
(i) Calculate the mean of the time until one of the two generators fails.
(ii) Calculate the mean of the time until both generators breaks down.
(iii) Calculate the probability that the smaller generators fails before the other one.

Solution: (i) Let X be the lifetime of the smaller generator. X has an exponential distribution with mean 20. Let Y be the lifetime of the larger generator. Y has an exponential distribution with mean 15. The mean of the time until one of the two generators fails is

$$E[\min(X, Y)] = \frac{1}{\frac{1}{20} + \frac{1}{15}} = \frac{60}{7}.$$

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(ii) $E[\max(X, Y)] = E[X + Y - \min(X, Y)] = 20 + 15 - \frac{60}{7} = \frac{185}{7}$.

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$$E[\min(X,Y)] = \frac{1}{\frac{1}{20}+\frac{1}{15}} = \frac{60}{7}.$$

(ii) $E[\max(X,Y)] = E[X+Y-\min(X,Y)] = 20+15-\frac{60}{7} = \frac{185}{7}$.
(iii) $\mathbb{P}\{X<Y\} = \frac{\frac{1}{20}}{\frac{1}{20}+\frac{1}{15}} = \frac{3}{7}$.

