Manual for SOA Exam MLC.

Chapter 10. Poisson processes.
Section 10.2. Poisson processes.

Extract from:
"Arcones' Manual for SOA Exam MLC. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Poisson processes

Definition 1

A stochastic process $\{N(t) : t \geq 0\}$ *is said to be a* **counting process** *if* $N(t)$ *represents the total number of "events" that have occurred up to time* t.

A counting process $N(t)$ must satisfy:
(i) $N(t) \geq 0$.
(ii) $N(t)$ is integer valued.
(iii) If $s < t$, then $N(s) \leq N(t)$.

For a counting process $\{N(t) : t \geq 0\}$ and $s < t$, $N(t) - N(s)$ is the number of events occurring in the time interval $(s, t]$.

Definition 2

A counting process is said to possess **independent increments** *if for each* $0 \leq t_1 < t_2 < \cdots < t_m$, $N(t_1), N(t_2) - N(t_1), N(t_3) - N(t_2), \ldots, N(t_m) - N(t_{m-1})$ *are independent r.v.'s.*

Notice that if $s < t$, $N(t) - N(s)$ is the increment of the process in the interval $[s, t]$.

Definition 3

A Poisson process is said to have **stationary increments** *if for each* $0 \leq t_1 \leq t_2$, $N(t_2) - N(t_1)$ *and* $N(t_2 - t_1) - N(0)$ *have the same distribution.*

In other words, a counting process has stationary increments if the distribution of an increment depends on its length, independently on its starting time.

Definition 4

An stochastic process $\{N(t) : t \geq 0\}$ *is said to be a* **Poisson process** *with rate* $\lambda > 0$*, if:*

(i) $N(0) = 0$.

(ii) The process has independent increments.

(iii) For each $0 \leq s, t$, $N(s + t) - N(s)$ *has a Poisson distribution with mean* λt.

Definition 4

An stochastic process $\{N(t) : t \geq 0\}$ *is said to be a* **Poisson process** *with rate* $\lambda > 0$*, if:*
(i) $N(0) = 0$*.*
(ii) The process has independent increments.
(iii) For each $0 \leq s, t$*,* $N(s + t) - N(s)$ *has a Poisson distribution with mean* λt*.*

Condition (iii) implies that a Poisson process has stationary increments.
In the previous definition, we may interpret $N(t)$ as the number of occurrences until time.
The rate of occurrences per unit of time is a constant. The average number of occurrences in the time interval $(s, s + t]$ is λt.

Theorem 1

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda > 0$. Then, for each $0 \leq t_1 < t_2 < \cdots < t_m$ and each $0 \leq k_1 \leq k_2 \leq \cdots \leq k_m$,

$$\mathbb{P}\{N(t_1) = k_1, N(t_2) = k_2, \dots, N(t_m) = k_m\}$$
$$= \frac{e^{-\lambda t_1} (\lambda t_1)^{k_1}}{k_1!} \frac{e^{-\lambda(t_2-t_1)} (\lambda(t_2 - t_1))^{k_2-k_1}}{(k_2 - k_1)!} \cdots$$
$$\cdots \frac{e^{-\lambda(t_m-t_{m-1})} (\lambda(t_m - t_{m-1}))^{k_m}}{k_m!}$$

Proof:

$$
\begin{aligned}
&\mathbb{P}\{N(t_1) = k_1, N(t_2) = k_2, \dots, N(t_m) = k_m\} \\
=&\mathbb{P}\{N(t_1) = k_1, N(t_2) - N(t_1) = k_2 - k_1, \dots \\
&\dots, N(t_m) - N(t_{m-1}) = k_m - k_{m-1}\} \\
=&\mathbb{P}\{N(t_1) = k_1\}\mathbb{P}\{N(t_2) - N(t_1) = k_2 - k_1\} \cdots \\
&\cdots \mathbb{P}\{N(t_m) - N(t_{m-1}) = k_m - k_{m-1}\} \\
=&e^{-\lambda t_1} \frac{(\lambda t_1)^{k_1}}{k_1!} e^{-\lambda(t_2 - t_1)} \frac{(\lambda(t_2 - t_1))^{k_2 - k_1}}{(k_2 - k_1)!} \cdots \\
&\cdots e^{-\lambda(t_m - t_{m-1})} \frac{(\lambda(t_m - t_{m-1}))^{k_m}}{k_m!}.
\end{aligned}
$$

Example 1

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$. *Compute:*
(i) $\mathbb{P}\{N(5) = 4\}$.
(ii) $\mathbb{P}\{N(5) = 4, N(6) = 9\}$.
(iii) $\mathbb{P}\{N(5) = 4, N(6) = 9, N(10) = 15\}$.
(iv) $\mathbb{P}\{N(5) - N(2) = 3\}$.
(v) $\mathbb{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$.
(vi) $\mathbb{P}\{N(2) + N(5) = 4\}$.

Example 1

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(v) $\mathbb{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$.
(vi) $\mathbb{P}\{N(2) + N(5) = 4\}$.

Solution:

(i)

$$\mathbb{P}\{N(5) = 4\} = \mathbb{P}\{\mathrm{Poiss}(10) = 4\} = \frac{e^{-10}(10)^4}{4!}.$$

Example 1

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$. *Compute:*
(i) $\mathbb{P}\{N(5) = 4\}$.
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(iii) $\mathbb{P}\{N(5) = 4, N(6) = 9, N(10) = 15\}$.
(iv) $\mathbb{P}\{N(5) - N(2) = 3\}$.
(v) $\mathbb{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$.
(vi) $\mathbb{P}\{N(2) + N(5) = 4\}$.

Solution:
(ii)

$$\begin{aligned}&\mathbb{P}\{N(5) = 4, N(6) = 9\} = \mathbb{P}\{N(5) = 4, N(6) - N(5) = 5\}\\ =&\mathbb{P}\{N(5) = 4\}\mathbb{P}(N(6) - N(5) = 5\}\\ =&\mathbb{P}\{\mathrm{Poiss}(10) = 4\}\mathbb{P}\{\mathrm{Poiss}(2) = 5\} = \frac{e^{-10}(10)^4}{4!}\frac{e^{-2}(2)^5}{5!}.\end{aligned}$$

Example 1

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$*. Compute:*
(i) $\mathbb{P}\{N(5) = 4\}$.
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(iv) $\mathbb{P}\{N(5) - N(2) = 3\}$.
(v) $\mathbb{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$.
(vi) $\mathbb{P}\{N(2) + N(5) = 4\}$.

Solution:

(iii)

$$\begin{aligned} &\mathbb{P}\{N(5) = 4, N(6) = 9, N(10) = 15\} \\ =&\mathbb{P}\{N(5) = 4, N(6) - N(5) = 5, N(10) - N(6) = 6\} \\ =&\mathbb{P}\{N(5) = 4\}\mathbb{P}\{N(6) - N(5) = 5\}\mathbb{P}\{N(10) - N(6) = 6\} \\ =&e^{-10}\frac{10^4}{4!}e^{-2}\frac{2^5}{5!}e^{-8}\frac{8^6}{6!}. \end{aligned}$$

Example 1

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(iii) $\mathbb{P}\{N(5) = 4, N(6) = 9, N(10) = 15\}$.
(iv) $\mathbb{P}\{N(5) - N(2) = 3\}$.
(v) $\mathbb{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$.
(vi) $\mathbb{P}\{N(2) + N(5) = 4\}$.

Solution:

(iv)

$$\mathbb{P}\{N(5) - N(2) = 3\} = e^{-6}\frac{6^3}{3!}.$$

Example 1

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$. *Compute:*
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(v) $\mathbb{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$.
(vi) $\mathbb{P}\{N(2) + N(5) = 4\}$.

Solution:

(v)

$$\begin{aligned}&\mathbb{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}\\ =&\mathbb{P}\{N(5) - N(2) = 3\}\mathbb{P}\{N(7) - N(6) = 4\} = e^{-6}\frac{6^3}{3!}e^{-2}\frac{2^4}{4!}.\end{aligned}$$

Example 1

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$. *Compute:*
(i) $\mathbb{P}\{N(5) = 4\}$.
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(v) $\mathbb{P}\{N(5) - N(2) = 3, N(7) - N(6) = 4\}$.
(vi) $\mathbb{P}\{N(2) + N(5) = 4\}$.

Solution:

(vi)

$$\begin{aligned}
&\mathbb{P}\{N(2) + N(5) = 4\} = \mathbb{P}\{2N(2) + (N(5) - N(2)) = 4\} \\
=&\mathbb{P}\{N(2) = 0, N(5) - N(2) = 4\} + \mathbb{P}\{N(2) = 1, N(5) - N(2) = 2\} \\
&+ \mathbb{P}\{N(2) = 2, N(5) - N(2) = 0\} \\
=&e^{-4} e^{-4} \frac{4^4}{4!} + e^{-4} \frac{4^1}{1!} e^{-4} \frac{4^2}{2!} + e^{-4} \frac{4^2}{2!} e^{-4}.
\end{aligned}$$

Theorem 2

For each $t \geq 0$,

$$E[N(t)] = \lambda t \text{ and } \mathrm{Var}(N(t)) = \lambda t.$$

Proof.

$N(t)$ has a Poisson distribution with mean λt. □

Theorem 3

For each $0 \leq s \leq t$,

$$\mathrm{Cov}(N(s), N(t)) = \lambda s.$$

Proof.

Since $N(s)$ and $N(t) - N(s)$ are independent, $\mathrm{Cov}(N(s), N(t) - N(s)) = 0$. So,

$$\begin{aligned} &\mathrm{Cov}(N(s), N(t)) = \mathrm{Cov}(N(s), N(s) + N(t) - N(s)) \\ =&\mathrm{Cov}(N(s), N(s)) + \mathrm{Cov}(N(s), N(t) - N(s)) = \mathrm{Var}(N(s)) = \lambda s, \end{aligned}$$

□

Example 2

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$. *Compute:*
(i) $E[2N(3) - 4N(5)]$.
(ii) $\mathrm{Var}(2N(3) - 4N(5))$.
(iii) $E[N(5) - 2N(6) + 3N(10)]$.
(iv) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$.
(v) $\mathrm{Cov}(N(5) - 2N(6), 3N(10))$.

Example 2

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$. *Compute:*
(i) $E[2N(3) - 4N(5)]$.
(ii) $\mathrm{Var}(2N(3) - 4N(5))$.
(iii) $E[N(5) - 2N(6) + 3N(10)]$.
(iv) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$.
(v) $\mathrm{Cov}(N(5) - 2N(6), 3N(10))$.

Solution:

(i)

$$E[2N(3) - 4N(5)] = 2E[N(3)] - 4E[N(5)]$$
$$=(2)(2)(3) - (4)(2)(5) = -28.$$

Example 2

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$*. Compute:*
(i) $E[2N(3) - 4N(5)]$.
(ii) $\mathrm{Var}(2N(3) - 4N(5))$.
(iii) $E[N(5) - 2N(6) + 3N(10)]$.
(iv) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$.
(v) $\mathrm{Cov}(N(5) - 2N(6), 3N(10))$.

Solution:

(ii)

$$\begin{aligned}&\mathrm{Var}(2N(3) - 4N(5)) = \mathrm{Var}(-2N(3) - 4(N(5) - N(3)))\\ =&(-2)^2\mathrm{Var}(N(3)) + (-4)^2\mathrm{Var}(N(5) - N(3))\\ =&(-2)^2(2)(3) + (-4)^2(2)(5-3) = 88.\end{aligned}$$

Example 2

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$. *Compute:*
(i) $E[2N(3) - 4N(5)]$.
(ii) $\mathrm{Var}(2N(3) - 4N(5))$.
(iii) $E[N(5) - 2N(6) + 3N(10)]$.
(iv) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$.
(v) $\mathrm{Cov}(N(5) - 2N(6), 3N(10))$.

Solution:
(iii)

$$E[N(5) - 2N(6) + 3N(10)] = (5)(2) - (2)(6)(2) + (3)(10)(2)$$
$$= 10 - 24 + 60 = 46.$$

Example 2

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 2$. Compute:
(i) $E[2N(3) - 4N(5)]$.
(ii) $\mathrm{Var}(2N(3) - 4N(5))$.
(iii) $E[N(5) - 2N(6) + 3N(10)]$.
(iv) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$.
(v) $\mathrm{Cov}(N(5) - 2N(6), 3N(10))$.

Solution:

(iv)

$$\begin{aligned}
&\mathrm{Var}(N(5) - 2N(6) + 3N(10)) \\
=&\mathrm{Var}(2N(5) + (N(6) - N(5)) + 3(N(10) - N(6))) \\
=&4\mathrm{Var}(N(5)) + \mathrm{Var}(N(1)) + 9\mathrm{Var}(N(4)) \\
=&(4)(5)(2) + (2)(1) + (9)(4)(2) = 114.
\end{aligned}$$

Example 2

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 2$*. Compute:*
(i) $E[2N(3) - 4N(5)]$.
(ii) $\mathrm{Var}(2N(3) - 4N(5))$.
(iii) $E[N(5) - 2N(6) + 3N(10)]$.
(iv) $\mathrm{Var}(N(5) - 2N(6) + 3N(10))$.
(v) $\mathrm{Cov}(N(5) - 2N(6), 3N(10))$.

Solution:

(v)

$$\mathrm{Cov}(N(5) - 2N(6), 3N(10)) = (3)(5)(2) - (6)(6)(2) = 30 - 72 - 42.$$

Theorem 4

Let $\{N(t) : t \geq 0\}$ be a counting process such that:
(i) $N(0) = 0$.
(ii) The process has independent stationary increments.
(iii) $\mathbb{P}\{N(h) \geq 2\} = o(h)$.
(iv) $\mathbb{P}\{N(h) = 1\} = \lambda h + o(h)$, where $\lambda > 0$.
Then, $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$.
Reciprocally, a Poisson process $\{N(t) : t \geq 0\}$ with rate $\lambda > 0$ is a counting process satisfying (i)–(iv).

Proof: See Arcones's manual.

Theorem 5

(Markov property of the Poisson process) Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* λ*. Let* $0 \leq t_1 < t_2 < \cdots < t_m < s$ *and let* $k_1 \leq k_2 \leq \cdots \leq k_m \leq j$*. Then,*

$$\begin{aligned} &\mathbb{P}\{N(s) = j | N(t_1) = k_1, \ldots, N(t_m) = k_m\} \\ =&\mathbb{P}\{N(s) = j | N(t_m) = k_m\}. \end{aligned}$$

Previous theorem says that a Poisson process is a Markov chain with continuous time and state space $E = \{0, 1, \ldots\}$.

Proof.

Since $N(t_1), N(t_2) - N(t_1), \cdots, N(t_m) - N(t_{m-1}), N(s) - N(t_m)$ are independent,

$$\begin{aligned}
&\mathbb{P}\{N(s) = j | N(t_1) = k_1, \dots, N(t_m) = k_m\} \\
&= \frac{\mathbb{P}\{N(t_1)=k_1, \dots, N(t_m)=k_m, N(s)=j\}}{\mathbb{P}\{N(t_1)=k_1, \dots, N(t_m)=k_m\}} \\
&= \frac{\mathbb{P}\{N(t_1)=k_1, N(t_2)-N(t_1)=k_2-k_1, \dots, N(t_m)-N(t_{m-1})=k_m-k_{m-1}, N(s)-N(t_m)=j-k_m\}}{\mathbb{P}\{N(t_1)=k_1, N(t_2)-N(t_1)=k_2-k_1, \dots, N(t_m)-N(t_{m-1})=k_m-k_{m-1}\}} \\
&= \frac{\mathbb{P}\{N(t_1)=k_1\} P\{N(t_2)-N(t_1)=k_2-k_1\} \cdots \mathbb{P}\{N(t_m)-N(t_{m-1})=k_m-k_{m-1}\} \mathbb{P}\{N(s)-N(t_m)=j-k_m\}}{\mathbb{P}\{N(t_1)=k_1\} P\{N(t_2)-N(t_1)=k_2-k_1\} \cdots \mathbb{P}\{N(t_m)-N(t_{m-1})=k_m-k_{m-1}\}} \\
&= \mathbb{P}\{N(s) - N(t_m) = j - k_m\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\{N(s) = j | N(t_m) = k_m\} = \mathbb{P}\{N(s) - N(t_m) = j - k_m | N(t_m) = k_m\} \\
&= \mathbb{P}\{N(s) - N(t_m) = j - k_m\}
\end{aligned}$$

□

Previous theorem implies that for $0 \le t_1 < t_2 < \cdots < t_m < s_1 < s_2 < \cdots s_m$ and for $k_1 \le k_2 \le \cdots \le k_m \le j_1 \le \cdots j_n$,

$$\begin{aligned}&\mathbb{P}\{N(s_1) = j_1, \dots, N(s_n) = j_n | N(t_1) = k_1, \dots, N(t_m) = k_m\}\\=&\mathbb{P}\{N(s_1) = j_1, \dots, N(s_n) = j_n | N(t_m) = k_m\}\end{aligned}$$

Theorem 6

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $t_0 > 0$ and let $j \geq 0$. Then, the distribution of $\{N(t) - N(t_0) : t \geq t_0\}$ conditional on $N(t_0) = j$ is that of a Poisson process with rate λ. In particular, for each $t_0 < s_1 < \cdots < s_n$ and each $j \leq k_1 \leq \ldots, k_m$,

$$
\begin{aligned}
&\mathbb{P}\{N(s_1) = k_1, \ldots, N(s_m) = k_m | N(t_0) = j\} \\
=&\mathbb{P}\{N(s_1 - t_0) = k_1 - j, \ldots, N(s_m - t_0) = k_m - j\}.
\end{aligned}
$$

Proof: Since a Poisson process has independent stationary increments,

$$\begin{aligned}
&\mathbb{P}\{N(s_1) = k_1, \dots, N(s_m) = k_m | N(t_0) = j\} \\
=&\mathbb{P}\{N(s_1) - N(t_0) = k_1 - j, N(s_2) - N(s_1) = k_m - k_{m-1}, \dots, \\
&\quad N(s_m) - N(s_{m-1}) = k_m - k_{m-1} | N(t_0) = j\} \\
=&\mathbb{P}\{N(s_1) - N(t_0) = k_1 - j\}\mathbb{P}\{N(s_2) - N(s_1) = k_2 - k_1\} \cdots \\
&\mathbb{P}\{N(s_m) - N(s_{m-1}) = k_m - k_{m-1}\} \\
=&\mathbb{P}\{\text{Pois}(\lambda(s_1 - t_0)) = k_1 - j\}\mathbb{P}\{\text{Pois}(\lambda(s_2 - s_1)) = k_2 - k_1\} \cdots \\
&\mathbb{P}\{\text{Pois}(\lambda(s_m - s_{m-1})) = k_m - k_{m-1}\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\{N(s_1 - t_0) = k_1 - j, \dots, N(s_m - t_0) = k_m - j\} \\
=&\mathbb{P}\{N(s_1 - t_0) = k_1 - j, N(s_2 - t_0) - N(s_1 - t_0) = k_2 - k_1 \dots, \\
&\quad N(s_m - t_0) - N(s_{m-1} - t_0) = k_m - k_{m-1}\} \\
=&\mathbb{P}\{N(s_1 - t_0) = k_1 - j\}\mathbb{P}\{N(s_2 - t_0) - N(s_1 - t_0) = k_2 - k_1\} \cdots \\
&\quad \mathbb{P}\{N(s_m - t_0) - N(s_{m-1} - t_0) = k_m - k_{m-1}\} \\
=&\mathbb{P}\{\text{Pois}(\lambda(s_1 - t_0)) = k_1 - j\}\mathbb{P}\{\text{Pois}(\lambda(s_2 - s_1)) = k_2 - k_1\} \cdots \\
&\mathbb{P}\{\text{Pois}(\lambda(s_m - s_{m-1})) = k_m - k_{m-1}\}.
\end{aligned}$$

It follows from the previous theorem that the distribution of $N(s+t)$ given $N(s)=j$ is that $j+\mathrm{Poisson}(\lambda t)$. So,

$$E[N(s+t)|N(s)=j]=j+\lambda t \text{ and } \mathrm{Var}(N(s+t)|N(s)=j)=\lambda t.$$

Previous theorem says that the number of occurrences from one moment on is a Poisson process. In some sense, the process starts anew at every time. Given a particular time, future occurrences from that time on follow a Poisson process with the same rate as the original process.

Example 3

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$. *Compute:*
(i) $\mathbb{P}\{N(5) = 7 | N(3) = 2\}$.
(ii) $E[2N(5) - 3N(7) | N(3) = 2]$.
(iii) $\mathrm{Var}(N(5) | N(2) = 3)$.
(iv) $\mathrm{Var}(N(5) - N(2) | N(2) = 3)$.
(v) $\mathrm{Var}(2N(5) - 3N(7) | N(3) = 2)$.

Example 3

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$. *Compute:*
(i) $\mathbb{P}\{N(5) = 7|N(3) = 2\}$.
(ii) $E[2N(5) - 3N(7)|N(3) = 2]$.
(iii) $\mathrm{Var}(N(5)|N(2) = 3)$.
(iv) $\mathrm{Var}(N(5) - N(2)|N(2) = 3)$.
(v) $\mathrm{Var}(2N(5) - 3N(7)|N(3) = 2)$.

Solution:

(i) $\mathbb{P}\{N(5) = 7|N(3) = 2\} = \mathbb{P}\{N(5) - N(3) = 7 - 2|N(3) = 2\} = \mathbb{P}\{N(2) = 5\} = e^{-6}\frac{6^5}{5!}$.

Example 3

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$. *Compute:*
(i) $\mathbb{P}\{N(5) = 7 | N(3) = 2\}$.
(ii) $E[2N(5) - 3N(7) | N(3) = 2]$.
(iii) $\mathrm{Var}(N(5) | N(2) = 3)$.
(iv) $\mathrm{Var}(N(5) - N(2) | N(2) = 3)$.
(v) $\mathrm{Var}(2N(5) - 3N(7) | N(3) = 2)$.

Solution:
(ii) $E[2N(5) - 3N(7) | N(3) = 2] = (2)(2+(3)(2)) - (3)(2+(3)(4)) = -26$.

Example 3

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$*. Compute:*
(i) $\mathbb{P}\{N(5) = 7 | N(3) = 2\}$.
(ii) $E[2N(5) - 3N(7) | N(3) = 2]$.
(iii) $\mathrm{Var}(N(5) | N(2) = 3)$.
(iv) $\mathrm{Var}(N(5) - N(2) | N(2) = 3)$.
(v) $\mathrm{Var}(2N(5) - 3N(7) | N(3) = 2)$.

Solution:
(iii)

$$\mathrm{Var}(N(5)|N(2) = 3) = \mathrm{Var}(N(5) - N(2) + 3|N(2) = 3)$$
$$= \mathrm{Var}(N(5) - N(2)|N(2) = 3) = \mathrm{Var}(N(3)) = (3)(3) = 9.$$

Example 3

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$. *Compute:*
(i) $\mathbb{P}\{N(5) = 7 | N(3) = 2\}$.
(ii) $E[2N(5) - 3N(7) | N(3) = 2]$.
(iii) $\mathrm{Var}(N(5) | N(2) = 3)$.
(iv) $\mathrm{Var}(N(5) - N(2) | N(2) = 3)$.
(v) $\mathrm{Var}(2N(5) - 3N(7) | N(3) = 2)$.

Solution:

(iv) $\mathrm{Var}(N(5)-N(2)|N(2) = 3) = \mathrm{Var}(N(5)-N(2)) = (3)(5-2) = 9$.

Example 3

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(iv) $\mathrm{Var}(N(5) - N(2) | N(2) = 3)$.
(v) $\mathrm{Var}(2N(5) - 3N(7) | N(3) = 2)$.

Solution:
(v)

$$\begin{aligned}
&\mathrm{Var}(2N(5) - 3N(7)|N(3) = 2) = \mathrm{Var}(2(N(2) + 2) - 3(N(4) + 2)) \\
=&\mathrm{Var}(2N(2) - 3N(4)) = \mathrm{Var}(-N(2) - 3(N(4) - N(2))) \\
=&\mathrm{Var}(-N(2)) + \mathrm{Var}(3(N(4) - N(2))) = (2)(3) + (3)^2(3)(4 - 2) = 60.
\end{aligned}$$

Theorem 7

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $s, t \geq 0$. Then,

$$P\{N(t) = k | N(s+t) = n\} = \binom{n}{k} \left(\frac{t}{t+s}\right)^k \left(\frac{s}{t+s}\right)^{n-k},$$

i.e. the distribution of $N(t)$ given $N(s+t) = n$ is binomial with parameters n and $p = \frac{t}{t+s}$. So,

$$E[N(t) | N(s+t) = n] = \frac{nt}{s+t}$$

and

$$\mathrm{Var}(N(t) | N(s+t) = n) = n \frac{t}{s+t} \frac{s}{s+t}.$$

Proof:

$$P\{N(t) = k | N(s+t) = n\} = \frac{\mathbb{P}\{N(t) = k, N(s+t) = n\}}{\mathbb{P}\{N(s+t) = n\}}$$

$$= \frac{\mathbb{P}\{N(t) = k, N(s+t) - N(t) = n-k\}}{\mathbb{P}\{N(s+t) = n\}}$$

$$= \frac{e^{-\lambda t}\frac{(\lambda t)^k}{k!} e^{-\lambda s}\frac{(\lambda s)^{n-k}}{(n-k)!}}{e^{-\lambda(s+t)}\frac{(\lambda(s+t))^n}{n!}} = \binom{n}{k}\left(\frac{t}{t+s}\right)^k \left(\frac{s}{t+s}\right)^{n-k}.$$

Previous theorem says that knowing that n events are recorded until time $s+t$, each of these events is recorded before time t with probability $\frac{t}{s+t}$ independently of the rest of events.

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Previous theorem can be extended as follows, given $0 \leq t_1 < t_2 < \cdots < t_m$, the conditional distribution of $(N(t_1), N(t_2) - N(t_1), \dots, N(t_m) - N(t_{m-1}))$ given $N(t_m) = n$ is multinomial distribution with parameter $(\frac{t_1}{t_m}, \frac{t_2 - t_1}{t_m}, \dots, \frac{t_m - t_{m-1}}{t_m})$.
Given $N(t_m) = n$, we know that events happens in the interval $[0, t_m]$, each of these events happens independently and the probability that one of these events happens in particular interval is the fraction of the total length of this interval.

Example 4

Customers arrive at a store according to a Poisson process with a rate 40 *customers per hour. Assume that three customers arrived during the first 15 minutes. Calculate the probability that no customer arrived during the first five minutes.*

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Customers arrive at a store according to a Poisson process with a rate 40 customers per hour. Assume that three customers arrived during the first 15 minutes. Calculate the probability that no customer arrived during the first five minutes.

Solution: Let $N(t)$ be the number of customers arriving in the first t minutes. $N(t)$ is a Poisson process with rate $2/3$. We have that

$$\mathbb{P}\{N(5) = 0 | N(15) = 3\} = \binom{3}{0} \left(\frac{5}{15}\right)^0 \left(\frac{10}{15}\right)^3 = 0.2962962963.$$