Manual for SOA Exam MLC.

Chapter 11. Poisson processes.
Section 11.3. Interarrival times.

Extract from:
"Arcones' Manual for SOA Exam MLC. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Arrival time

For $n \geq 1$, let S_n be the arrival time of the n–th event, i.e.

$$S_n = \inf\{t \geq 0 : N(t) = n\}.$$

Notice that $N(S_n) = n$ and $N(t) < n$, for $t < S_n$.

An useful relation between $N(t)$ and S_n is

$$\begin{aligned}&\{S_n \le t\}\\ =&\{\text{the } n\text{−th occurrence happens before time } t\}\\ =&\{\text{there are } n \text{ or more occurrences in the interval } [0,t]\}\\ =&\{N(t) \ge n\}.\end{aligned}$$

Since $\{S_n \le t\} = \{N(t) \ge n\}$,

$$\{S_n > t\} = \{N(t) < n\}$$

and

$$\begin{aligned}&\{N(t) = n\} = \{N(t) \ge n\} \cap \{N(t) < n+1\}\\ =&\{S_n \le t\} \cap \{S_{n+1} > t\} = \{S_n \le t < S_{n+1}\}.\end{aligned}$$

Theorem 1

For each integer $n \geq 1$ and each $t \geq 0$,

$$\mathbb{P}\{\text{Gamma}(n,1) > t\} = \mathbb{P}\{\text{Poisson}(t) \leq n-1\}.$$

Proof.

Using that $\int \frac{x^n}{n!} e^{-x}\, dx = -e^{-x} \sum_{j=0}^{n} \frac{x^j}{j!} + c$,

$$\mathbb{P}\{\text{Gamma}(n,1) > t\} = \int_t^\infty \frac{x^n}{n!} e^{-x}\, dx = -e^{-x} \sum_{j=0}^{n} \frac{x^j}{j!} \Big|_t^\infty$$

$$= -\, e^{-t} \sum_{j=0}^{n} \frac{t^j}{j!} = \mathbb{P}\{\text{Poiss}(t) \leq n-1\}.$$

□

Theorem 2

S_n has a gamma distribution with parameters $\alpha = n$ and $\beta = \frac{1}{\lambda}$.

Proof.

By Theorem 1,

$$\begin{aligned}&\mathbb{P}\{S_n > t\} = \mathbb{P}\{N(t) < n\} = \mathbb{P}\{\mathrm{Poiss}(\lambda t) \le n-1\}\\ =&\mathbb{P}\{\mathrm{Gamma}(n,1) > \lambda t\} = \mathbb{P}\left\{\mathrm{Gamma}\left(n, \frac{1}{\lambda}\right) > t\right\}.\end{aligned}$$

□

Since S_n has a gamma distribution with parameters $\alpha = n$ and $\beta = \frac{1}{\lambda}$, S_n has density function

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, t \geq 0,$$

$E[S_n] = \frac{n}{\lambda}$ and $\mathrm{Var}(S_n) = \frac{n}{\lambda^2}$.
The c.d.f. of S_n is

$$F_{S_n}(t) = \mathbb{P}\{S_n \leq t\} = \mathbb{P}\{N(t) \geq n\} = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}.$$

Example 1

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$*. Let* S_n *denote the time of the occurrence of the n–th event. Calculate:*
(i) $\mathbb{P}\{S_3 > 5\}$.
(ii) The density of S_3.
(iii) Find the expected value and the variance of S_3.
(iv) $\mathbb{P}\{S_2 > 3, S_5 > 7\}$.

Example 1

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Calculate:
(i) $\mathbb{P}\{S_3 > 5\}$.
(ii) The density of S_3.
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(iv) $\mathbb{P}\{S_2 > 3, S_5 > 7\}$.

Solution:
(i)

$$\mathbb{P}\{S_3 > 5\} = \mathbb{P}\{N(5) \leq 2\} = e^{-15}\left(1 + 15 + \frac{15^2}{2}\right)$$

$=0.00003930844818.$

Example 1

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(i) $\mathbb{P}\{S_3 > 5\}$.
(ii) The density of S_3.
(iii) Find the expected value and the variance of S_3.
(iv) $\mathbb{P}\{S_2 > 3, S_5 > 7\}$.

Solution:

(ii) S_3 has a gamma distribution with $\alpha = 3$ and $\beta = \frac{1}{3}$. Hence, the density of S_3 is

$$f_{S_3}(t) = \frac{3^3 t^2 e^{-3t}}{3!} = \frac{9t^2 e^{-3t}}{2}, t \geq 0.$$

Example 1

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Calculate:
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(ii) The density of S_3.
(iii) Find the expected value and the variance of S_3.
(iv) $\mathbb{P}\{S_2 > 3, S_5 > 7\}$.

Solution:
(iii)

$$E[S_3] = 3(1/3) = 1 \text{ and} \mathrm{Var}(S_3) = 3(1/3)^2 = 1/3.$$

(iv)

$$
\begin{aligned}
&\mathbb{P}\{S_2 > 3, S_5 > 7\} = \mathbb{P}\{N(3) \le 1, N(7) \le 4\} \\
=&\mathbb{P}\{N(3) = 0, N(7) \le 4\} + \mathbb{P}\{N(3) = 1, N(7) \le 4\} \\
=&\mathbb{P}\{N(3) = 0\}\mathbb{P}\{N(7) - N(3) \le 4\} \\
&+ \mathbb{P}\{N(3) = 1\}\mathbb{P}\{N(7) - N(3) \le 3\} \\
=&e^{-9}e^{-12}\left(1 + 12 + \frac{12^2}{2} + \frac{12^3}{6} + \frac{12^4}{24}\right) \\
&+ e^{-9}(9)e^{-12}\left(1 + 12 + \frac{12^2}{2} + \frac{12^3}{6}\right) \\
=&e^{-21}(1237) + e^{-21}(3357) = (3.483428261)10^{-6}.
\end{aligned}
$$

Theorem 3

The joint density of $(S_1, \dots, S_n)$ *is*

$$f_{S_1,\dots,S_n}(s_1, \dots, s_n) = \lambda^n e^{-\lambda s_n}, \text{ if } 0 < s_1 < s_2 < \cdots < s_n.$$

Proof: Given $0 < s_1 < s_2 < \cdots < s_n$, take $h > 0$ small enough. Then,

$$\begin{aligned}
&\mathbb{P}\{S_1 \in (s_1, s_1 + h], \dots, S_n \in (s_n, s_n + h]\}\\
=&\mathbb{P}\{N(s_1) = 0, N(s_1 + h) - N(s_1) = 1, N(s_2) - N(s_1 + h) = 0, \dots,\\
&N(s_n) - N(s_{n-1} + h) = 0, N(s_n + h) - N(s_n) = 1\}\\
=&e^{-\lambda s_1} e^{-\lambda h} \lambda h e^{-\lambda(s_2 - s_1 - h)} \cdots e^{-\lambda(s_n - s_{n-1} - h)} e^{-\lambda h} \lambda h\\
=&\lambda^n h^n e^{-\lambda s_n}
\end{aligned}$$

$$\begin{aligned}
&f_{S_1,\dots,S_n}(s_1, \dots, s_n)\\
=&\lim_{h \to 0+} \frac{\mathbb{P}\{S_1 \in (s_1, s_1 + h], \dots, S_n \in (s_n, s_n + h]\}}{h^n} = \lambda^n e^{-\lambda s_n}.
\end{aligned}$$

The distribution of $(S_1, \dots, S_{n-1})$ given S_n is uniform on $0 < s_1 < s_2 < \cdots < s_n$, i.e.

$$f_{S_1,\dots,S_{n-1}|S_n}(s_1, \dots, s_{n-1}|s_n) = \frac{f_{S_1,\dots,S_n}(s_1, \dots, s_n)}{f_{S_n}(s_n)}$$

$$= \frac{\lambda^n e^{-\lambda s_n}}{\frac{\lambda^n s_n^{n-1} e^{-\lambda s_n}}{(n-1)!}} = \frac{(n-1)!}{s_n^{n-1}}, \text{ for } 0 < s_1 < s_2 < \cdots < s_n.$$

Since a Poisson process is a stationary process, we should expect this distribution.

Let $T_n = S_n - S_{n-1}$ be the time elapsed between the $(n-1)$–th and the n–th event. T_n is called the **interarrival** between the $(n-1)$–th and the n–th event.

Theorem 4

$\{T_n\}_{n=1}^{\infty}$, are independent identically distributed exponential random variables having mean $\frac{1}{\lambda}$.

Theorem 4 says that if the rate of events is λ events per unit of time, then the expected waiting time between events is $\frac{1}{\lambda}$. Theorem 4 implies that $E[T_n] = \frac{1}{\lambda}$, $\mathrm{Var}(T_n) = \frac{1}{\lambda^2}$ and T_n has density function

$$f_{T_n}(t) = \lambda e^{-\lambda t}, t \geq 0.$$

Theorem 4 implies that for $k_1 < k_2 < \cdots < k_m$,

$$S_{k_1}, S_{k_2} - S_{k_1}, S_{k_m} - S_{k_{m-1}}$$

are independent r.v.'s.

Proof.

We have that $(S_1, \dots, S_n) = (T_1, T_1 + T_2, \dots, T_1 + \cdots + T_n)$. This transformation has Jacobian one. Hence, the density of $(T_1, \dots, T_n)$ is

$$f_{T_1,\dots,T_n}(t_1, \dots, t_n) = f_{S_1,\dots,S_n}(t_1, t_1 + t_2, \dots, t_1 + \cdots + t_n)$$
$$= \lambda^n e^{-\lambda(t_1+\cdots+t_n)} = \prod_{j=1}^{n} \lambda e^{-\lambda t_j}.$$

□

Given a sequence of independent identically distributed r.v.'s $\{T_n\}_{n=1}^{\infty}$ with an exponential distribution with mean $\frac{1}{\lambda}$, define

$$N(t) = \sup\{n \geq 0 : S_n \leq t\}, t \geq 0,$$

where $S_n = \sum_{j=1}^{n} T_j$, $n \geq 1$, $S_n = 0$. It is possible to prove that $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Hence, from a Poisson process we can obtain a sequence of independent identically distributed r.v.'s with an exponential distribution, and reciprocally from a sequence of independent identically distributed r.v.'s with an exponential distribution, we can obtain a Poisson process.

Example 2

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$. *Let* S_n *denote the time of the occurrence of the n–th event. Let* $T_n = S_n - S_{n-1}$ *be the elapsed time between the* $(n-1)$*–th and the n–th event. Calculate:*

(i) The density of T_6.

(ii) Find the expected value and the variance of T_6.

(iii) Find $\mathrm{Cov}(T_3, T_8)$.

(iv) Find $\mathrm{Cov}(S_2, S_9)$.

Example 2

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Let $T_n = S_n - S_{n-1}$ be the elapsed time between the $(n-1)$–th and the n–th event. Calculate:
(i) The density of T_6.
(ii) Find the expected value and the variance of T_6.
(iii) Find $\mathrm{Cov}(T_3, T_8)$.
(iv) Find $\mathrm{Cov}(S_2, S_9)$.

Solution:

(i) T_6 has an exponential distribution with mean $\frac{1}{3}$. Hence, the density of T_6 is

$$f_{T_6}(t) = 3e^{-3t}, t \geq 0.$$

Example 2

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Let $T_n = S_n - S_{n-1}$ be the elapsed time between the $(n-1)$–th and the n–th event. Calculate:

(i) The density of T_6.

(ii) Find the expected value and the variance of T_6.

(iii) Find $\mathrm{Cov}(T_3, T_8)$.

(iv) Find $\mathrm{Cov}(S_2, S_9)$.

Solution:

(ii)

$$E[T_6] = (1/3), \mathrm{Var}(T_6) = (1/3)^2 = 1/9.$$

Example 2

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Let $T_n = S_n - S_{n-1}$ be the elapsed time between the $(n-1)$–th and the n–th event. Calculate:

(i) The density of T_6.

(ii) Find the expected value and the variance of T_6.

(iii) Find $\mathrm{Cov}(T_3, T_8)$.

(iv) Find $\mathrm{Cov}(S_2, S_9)$.

Solution:

(iii) $\mathrm{Cov}(T_3, T_8) = 0$.

Example 2

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(i) The density of T_6.

(ii) Find the expected value and the variance of T_6.

(iii) Find $\mathrm{Cov}(T_3, T_8)$.

(iv) Find $\mathrm{Cov}(S_2, S_9)$.

Solution:

(iv)

$$\mathrm{Cov}(S_2, S_9) = \mathrm{Cov}(S_2, S_2 + S_9 - S_2) = \mathrm{Cov}(S_2, S_2) = 2(1/3)^2 = 2/9.$$

Theorem 5

Given $0 \leq n < m$ *and* $t > 0$,

$$\mathbb{P}\{S_m - t > s | N(t) = n\} = \mathbb{P}\{S_{m-n} > s\}, s > 0,$$

and

$$E[S_m | N(t) = n] = t + E[S_{n-m}].$$

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and

$$E[S_m | N(t) = n] = t + E[S_{n-m}].$$

Proof: We have that

$$\mathbb{P}\{S_m - t > s | N(t) = n\} = \mathbb{P}\{S_m > t + s | N(s) = n\}$$
$$= \mathbb{P}\{N(t+s) < m | N(t) = n\}$$
$$= \mathbb{P}\{N(t+s) - N(t) < m - n | N(t) = n\}$$
$$= \mathbb{P}\{N(t+s) - N(t) < m - n\} = \mathbb{P}\{N(s) < m - n\} = P\{S_{m-n} > s\}.$$

Theorem 5

Given $0 \le n < m$ *and* $t > 0$,

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and

$$E[S_m | N(t) = n] = t + E[S_{n-m}].$$

Proof:

$$\begin{aligned} E[S_m | N(t) = n] &= t + E[S_m - t | N(t) = n] \\ =& t + \int_0^\infty \mathbb{P}\{S_m - t > s | N(t) = n\} \, ds \\ =& t + \int_0^\infty \mathbb{P}\{S_{m-n} > s\} \, ds = t + E[S_{m-n}]. \end{aligned}$$

Theorem 5

Given $0 \leq n < m$ *and* $t > 0$,

$$\mathbb{P}\{S_m - t > s | N(t) = n\} = \mathbb{P}\{S_{m-n} > s\}, s > 0,$$

and

$$E[S_m | N(t) = n] = t + E[S_{n-m}].$$

A Poisson process is a Markov process. it starts anew. The number of occurrences observed after time t has the same distribution as the number of occurrences observed after time zero. Hence, given $N(t) = n$ the waiting time after time t until the m–th occurrence is observed has the distribution of the waiting time until the $(m-n)$–th occurrence is observed.

Example 3

Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$*. Let* S_n *denote the time of the occurrence of the n–th event. Let* $T_n = S_n - S_{n-1}$ *be the elapsed time between the* $(n-1)$*–th and the n–th event. Calculate:*

(i) $E[S_3|N(4) = 1]$.
(ii) $\mathbb{P}\{S_3 > 7|N(4) = 1\}$.
(iii) $\mathbb{P}\{T_3 > 7|N(4) = 1\}$.

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Let $\{N(t) : t \geq 0\}$ *be a Poisson process with rate* $\lambda = 3$. *Let* S_n *denote the time of the occurrence of the n–th event. Let* $T_n = S_n - S_{n-1}$ *be the elapsed time between the* $(n-1)$*–th and the n–th event. Calculate:*

(i) $E[S_3|N(4) = 1]$.

(ii) $\mathbb{P}\{S_3 > 7|N(4) = 1\}$.

(iii) $\mathbb{P}\{T_3 > 7|N(4) = 1\}$.

Solution:

(i) $E[S_3|N(4) = 1) = 4 + E[S_{3-1}] = 4 + (3-1)\frac{1}{3} = \frac{14}{3}$.

Example 3

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Let $T_n = S_n - S_{n-1}$ be the elapsed time between the $(n-1)$–th and the n–th event. Calculate:
(i) $E[S_3 | N(4) = 1]$.
(ii) $\mathbb{P}\{S_3 > 7 | N(4) = 1\}$.
(iii) $\mathbb{P}\{T_3 > 7 | N(4) = 1\}$.

Solution:
(ii)

$$\mathbb{P}\{S_3 > 7 | N(4) = 1\} = \mathbb{P}\{S_{3-1} > 7 - 4\} = \mathbb{P}\{S_2 > 3\}$$
$$= \mathbb{P}\{N(3) \leq 1\} = e^{-9}(1 + 9) = 0.001234.$$

Example 3

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Let $T_n = S_n - S_{n-1}$ be the elapsed time between the $(n-1)$–th and the n–th event. Calculate:
(i) $E[S_3|N(4) = 1]$.
(ii) $\mathbb{P}\{S_3 > 7|N(4) = 1\}$.
(iii) $\mathbb{P}\{T_3 > 7|N(4) = 1\}$.

Solution:
(ii) Alternatively,

$$\mathbb{P}\{S_3 > 7|N(4) = 1\} = \mathbb{P}\{N(7) \leq 2|N(4) = 1\} = \mathbb{P}\{N(3) \leq 1\}$$
$$= e^{-9}(1 + 9) = 0.001234.$$

Example 3

Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda = 3$. Let S_n denote the time of the occurrence of the n–th event. Let $T_n = S_n - S_{n-1}$ be the elapsed time between the $(n-1)$–th and the n–th event. Calculate:
(i) $E[S_3|N(4) = 1]$.
(ii) $\mathbb{P}\{S_3 > 7|N(4) = 1\}$.
(iii) $\mathbb{P}\{T_3 > 7|N(4) = 1\}$.

Solution:
(iii)

$$\mathbb{P}\{T_3 > 5|N(4) = 1\} = \mathbb{P}\{T_3 > 5|T_1 \leq 4 < T_1 + T_2\}$$
$$= \mathbb{P}\{T_3 > 5\} = e^{-15}.$$

Theorem 6

Given that $N(t) = n$, the n arrival times $S_1, \dots, S_n$ have conditional density

$$f_{S_1,\dots,S_n|N(t)=n}(s_1, \dots, s_n) = \frac{n!}{t^n}, 0 < s_1 < s_2 < \cdots < s_n \le t.$$

The proof of the previous theorem is in Arcones' manual.
It follows from the previous theorem that the distribution of $T_1, \dots, T_n$ given $N(t) = n$, is

$$f_{T_1,\dots,T_n|N(t)=n}(t_1, \dots, t_n) = \frac{n!}{t^n}, 0 < t_1, t_2, \dots, t_n, t_1 + t_2 + \cdots + t_n \le t.$$

It also follows from the previous theorem that the distribution of the n arrival times $S_1, \dots, S_n$ given $N(t) = n$, is the distribution of order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.

Theorem 7

Given $s < t$ and $k \le n$,

$$P\{N(s) = k | S_n = t\} = \binom{n-1}{k} \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^{n-k-1}.$$

Previous theorem says that given $S_n = t$, the number of occurrences in the interval $[0, s]$ has a binomial distribution with $n-1$ trials and success probability $\frac{s}{t}$.

Proof:

$$
\begin{aligned}
&\mathbb{P}\{N(s) = k | S_n = t\} \\
&= \lim_{\epsilon \to 0+} P\{N(s) = k | N(t) = n, N(t-\epsilon) = n-1\} \\
&= \lim_{\epsilon \to 0+} \frac{\mathbb{P}\{N(s) = k, N(t) = n, N(t-\epsilon) = n-1\}}{P\{N(t) = n, N(t-\epsilon) = n-1\}} \\
&= \lim_{\epsilon \to 0+} \frac{e^{-\lambda s}\frac{(\lambda s)^k}{k!} e^{-\lambda(t-\epsilon-s)} \frac{(\lambda(t-\epsilon-s))^{n-1-k}}{(n-1-k)!} e^{-\lambda\epsilon}\lambda\epsilon}{e^{-\lambda(t-\epsilon)}\frac{(\lambda(t-\epsilon))^{n-1}}{(n-1)!} e^{-\lambda\epsilon}\lambda\epsilon} \\
&= \binom{n-1}{k} \left(\frac{s}{t}\right)^k \left(\frac{t-s}{t}\right)^{n-k-1}.
\end{aligned}
$$

Example 4

Claims arrive to an insurance company website according with a Poisson rate of 100 *claims per day. Suppose that in one day the 10–th claim after midnight arrived at 5:15 am. Calculate the probability that more than two claims arrived between midnight and 1:00 am.*

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Claims arrive to an insurance company website according with a Poisson rate of 100 *claims per day. Suppose that in one day the 10–th claim after midnight arrived at 5:15 am. Calculate the probability that more than two claims arrived between midnight and 1:00 am.*

Solution: The conditional distribution is binomial with $n = 9$ and $p = \frac{60}{(5)(60)+15} = \frac{4}{21}$. Hence, the probability that more than two claims arrived between midnight and 1:00 am. is

$$1 - \binom{9}{0}\left(\frac{4}{21}\right)^0\left(\frac{17}{21}\right)^9 - \binom{9}{1}\left(\frac{4}{21}\right)^1\left(\frac{17}{21}\right)^8$$
$$=1 - 0.1493023487 - 0.3161696796 = 0.5345279717.$$

