Manual for SOA Exam MLC.

Chapter 11. Poisson processes.

Section 11.4. Superposition and decomposition of a Poisson process.

Extract from:
"Arcones' Manual for SOA Exam MLC. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Superposition and decomposition of a Poisson process

We know that:

Theorem 1

Let $X \sim \mathrm{Poisson}(\lambda_1)$ *and* $Y \sim \mathrm{Poisson}(\lambda_2)$. *Suppose that* X *and* Y *are independent. Then,* $X + Y \sim \mathrm{Poisson}(\lambda_1 + \lambda_2)$.

Next theorem generalizes previous theorem to Poisson processes.

Theorem 2

If $\{N_1(t) : t \geq 0\}$ *and* $\{N_2(t) : t \geq 0\}$ *are two independent Poisson processes with respective rates* λ_1 *and* λ_2. *Then,* $\{N_1(t) + N_2(t) : t \geq 0\}$ *is a Poisson process with rate* $\lambda_1 + \lambda_2$.

The Poisson process $\{N_1(t) + N_2(t) : t \geq 0\}$ is called the superposition of $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$.

Proof.

Let $N(t) = N_1(t) + N_2(t)$. Given $0 \le t_1 < t_2 < \cdots < t_m$,

$$N_1(t_1), N_1(t_2) - N_1(t_1), \ldots, N_1(t_m) - N_1(t_{m-1}),$$
$$N_2(t_1), N_2(t_2) - N_1(t_1), \ldots, N_2(t_m) - N_1(t_{m-1})$$

are independent r.v.'s. So,

$$N(t_1), N(t_2) - N(t_1), \ldots, N(t_m) - N(t_{m-1})$$

are independent r.v.'s. Besides,
$N_1(t_j) - N_1(t_{j-1}) \sim \mathrm{Poisson}(\lambda_1(t_j - t_{j-1}))$ and
$N_2(t_j) - N_2(t_{j-1}) \sim \mathrm{Poisson}(\lambda_2(t_j - t_{j-1}))$. So,
$N(t_j) - N(t_{j-1}) \sim \mathrm{Poisson}((\lambda_1 + \lambda_2)(t_j - t_{j-1}))$. Hence,
$\{N(t) : t \ge 0\}$ is a Poisson processes with rate $\lambda_1 + \lambda_2$. □

Example 1

An insurance company receives two type of claims: car and home. The number of car insurance claims received follows a Poisson process distribution with rate 20 claims per day. The number of home insurance claims received follows a Poisson process distribution with rate 5 claims per day. Both processes are independent. Estimate the probability that this insurance company receives more than a 800 claims in a 30 day period.

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Solution: The number of claims received until time t follows a Poisson process with rate 25 claims per day. We have that

$$E[N(30)] = (30)(25) = 750, \mathrm{Var}(N(30)) = (30)(25) = 750$$

$$\text{and } \mathbb{P}\{N(30) > 800\} \approx \mathbb{P}\left\{N(0,1) > \frac{800.5 - 750}{\sqrt{750}}\right\}$$

$$\approx \mathbb{P}\{N(0,1) > 1.84\} = 0.0329.$$

Theorem 3

Let $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ be two independent Poisson processes with respective rates λ_1 and λ_2. Let $N(t) = N_1(t) + N_2(t)$, $t \geq 0$. Let $\lambda = \lambda_1 + \lambda_2$. Then, the conditional distribution of $N_1(t)$ given $N(t) = n$ is binomial with parameters n and $p = \frac{\lambda_1}{\lambda}$.

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The previous theorem states that given that the total number of occurrences is n, the probability that a given occurrence is of type 1 is $\frac{\lambda_1}{\lambda}$ independently of the rest of the occurrences.

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Proof.

We have that

$$\mathbb{P}\{N_1(t) = k | N(t) = n\} = \frac{\mathbb{P}\{N_1(t)=k,\ N(t)=n\}}{\mathbb{P}\{N(t)=n\}} = \frac{\mathbb{P}\{N_1(t)=k,\ N_2(t)=n-k\}}{\mathbb{P}\{N(t)=n\}}$$

$$= \frac{\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \cdot \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} = \binom{n}{k} p^k (1-p)^{n-k}.$$

□

Example 2

An insurance company receives two type of claims: car and home. The number of car insurance claims received follows a Poisson process distribution with rate 20 claims per day. The number of home insurance claims received follows a Poisson process distribution with rate 5 claims per day. Both processes are independent. Suppose that in a given day five claims are received. Calculate the probability that exactly three claims are car insurance claims.

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Solution: Let X be the number of claims which are are car insurance claims. X has a binomial distribution with $n = 5$ and $p = \frac{20}{20+5} = \frac{4}{5}$. Hence,

$$\mathbb{P}[X = 3\} = \binom{5}{3} \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^2 = 0.2048.$$

Decomposition of a Poisson process.

Consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ. Suppose that each time an event occurs it is classified as either a type I or a type II event. Suppose further that each event is classified as type I event with probability p and as type II event with probability $1 - p$. Let $N_1(t)$ denote respectively the number of type I events occurring in $[0, t]$. Let $N_2(t)$ denote respectively the number of type II events occurring in $[0, t]$. Note that $N(t) = N_1(t) + N_2(t)$.

Theorem 4

$\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent Poisson processes with respective rates $\lambda_1 = \lambda p$ and $\lambda_2 = \lambda(1 - p)$.

Proof: See Arcones' manual.

Example 3

Customers arrive to a store according with a Poisson process with rate $\lambda = 20$ *arrivals per hour. Suppose that the probability that a customer buys something is* $p = 0.30$.

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(i) Find the expected number of sales made during an eight-hour business day.

Solution: (i) Let $N_1(t)$ be the number of arrivals who buy something. Let $N_2(t)$ be the number of arrivals who do not buy something. N_1 and N_2 are two independent Poisson processes. The rate for N_1 is $\lambda_1 = \lambda p = (20)(0.3) = 6$. Hence, $E[N_1(8)] = (8)(6) = 48$.

Example 3

Customers arrive to a store according with a Poisson process with rate $\lambda = 20$ arrivals per hour. Suppose that the probability that a customer buys something is $p = 0.30$.

(ii) Find the probability that 10 or more sales are made in a period of one hour.

Solution: (ii) $\mathbb{P}\{N_1(1) \geq 10\} = 1 - \sum_{j=0}^{9} \mathbb{P}\{N_1(1) = j\} = 1 - \sum_{j=0}^{9} \frac{e^{-6}6^j}{j!} = 0.04262092358.$

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Customers arrive to a store according with a Poisson process with rate $\lambda = 20$ arrivals per hour. Suppose that the probability that a customer buys something is $p = 0.30$.

(iii) The store opens at 8 a.m. find the expected time of the tenth sale of the day.

Solution: (iii) Let $S_{10,1}$ be the time of the tenth sale of the day. Then, $E[S_{10,1}] = (10)\frac{1}{\lambda_1} = \frac{10}{6}$ hours or 100 minutes. The expected time of the tenth sale is $9:40$ a.m.

Consider two independent Poisson processes $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ with respective rates λ_1 and λ_2. Let $S_{1,n}$ be the time of the n–th arrival for the first Poisson process. Let $S_{2,m}$ be the time of the m–th arrival for the second Poisson process. Suppose that $S_{1,n} < S_{2,m}$, then at the time of the $n + m - 1$–th arrival for both Poisson processes, we have observed n or more arrivals of the first Poisson process. Hence,

$$\begin{aligned}&\mathbb{P}\{S_{1,n} < S_{2,m}\}\\ =&\sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}.\end{aligned}$$

Example 4

An insurance company receives two type of claims: car and home. The number of car insurance claims received follows a Poisson process distribution with rate 20 claims per day. The number of home insurance claims received follows a Poisson process distribution with rate 5 claims per day. Both processes are independent. Calculate the probability that at least two car insurance claims arrive before three home insurance claims arrive.

Example 4

An insurance company receives two type of claims: car and home. The number of car insurance claims received follows a Poisson process distribution with rate 20 claims per day. The number of home insurance claims received follows a Poisson process distribution with rate 5 claims per day. Both processes are independent. Calculate the probability that at least two car insurance claims arrive before three home insurance claims arrive.

Solution: The probability that a claim is a car insurance is $\frac{20}{20+5} = \frac{4}{5}$. When at least two car insurance claims arrive before three home insurance claims arrive, we have that form the first fours claims, we get two or more car insurance claims. Hence, the answer is

$$\binom{4}{2}\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2 + \binom{4}{3}\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^1 + \binom{4}{4}\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^0$$

$$=0.1536 + 0.4096 + 0.4096 = 0.9728.$$

