

# Manual for SOA Exam MLC.

Chapter 11. Poisson processes.

Section 11.5. Nonhomogenous Poisson processes

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# Nonhomogenous Poisson processes

## Definition 1

The counting process  $\{N(t) : t \geq 0\}$  is said to be a **nonhomogenous Poisson process with intensity function  $\lambda(t)$** ,  $t \geq 0$ , if

(i)  $N(0) = 0$ .

(ii) For each  $t > 0$ ,  $N(t)$  has a Poisson distribution with mean  $m(t) = \int_0^t \lambda(s) ds$ .

(iii) For each  $0 \leq t_1 < t_2 < \dots < t_m$ ,  $N(t_1), N(t_2) - N(t_1), \dots, N(t_m) - N(t_{m-1})$  are independent r.v.'s.

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- ▶ An nonhomogeneous Poisson process with  $\lambda(t) = \lambda$ , for each  $t \geq 0$ , is a regular Poisson process.
- ▶  $m(t)$  is the **mean value function** of the non homogeneous Poisson process.

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- ▶ For each  $0 \leq t_1 < t_2 < \dots < t_m$  and each integers  $k_1, \dots, k_m \geq 0$ ,

$$\begin{aligned}
 & \mathbb{P}\{N(t_1) = k_1, N(t_2) = k_2, \dots, N(t_m) = k_m\} \\
 = & \mathbb{P}\{N(t_1) = k_1, N(t_2) - N(t_1) = k_2 - k_1, \dots, \\
 & N(t_m) - N(t_{m-1}) = k_m - k_{m-1}\} \\
 = & \frac{e^{-m(t_1)}(m(t_1))^{k_1}}{k_1!} \frac{e^{-(m(t_2)-m(t_1))}(m(t_2)-m(t_1))^{k_2-k_1}}{(k_2-k_1)!} \dots \\
 & \frac{e^{-(m(t_m)-m(t_{m-1}))}(m(t_m)-m(t_{m-1}))^{k_m-k_{m-1}}}{(k_m-k_{m-1})!},
 \end{aligned}$$



## Example 1

*For a nonhomogenous Poisson process the intensity function is given by*

$$\lambda(t) = \begin{cases} 5 & \text{if } t \text{ is in } (1, 2], (3, 4], \dots \\ 3 & \text{if } t \text{ is in } (0, 1], (2, 3], \dots \end{cases}$$

*Find the probability that the number number of observed occurrences in the time period  $(1.25, 3]$  is more than two.*

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Find the probability that the number number of observed occurrences in the time period  $(1.25, 3]$  is more than two.

**Solution:**  $N(3) - N(1.25)$  has a Poisson distribution with mean

$$m(3) - m(1.25) = \int_{1.25}^3 \lambda(t) dt = \int_{1.25}^2 5 dt + \int_2^3 3 dt = 6.75.$$

Hence,

$$\begin{aligned} \mathbb{P}\{N(3) - N(1.25) > 2\} &= 1 - e^{-6.75}(1 + 6.75 + (6.75)^2/2) \\ &= 0.9642515816. \end{aligned}$$

Let  $S_n$  be the time of the  $n$ -th occurrence.  $S_n$  is an extended r.v. with values in  $[0, \infty]$ . Then,

$$\begin{aligned}\mathbb{P}\{S_n > t\} &= \mathbb{P}\{N(t) \leq n-1\} = \sum_{j=0}^{n-1} \frac{e^{-m(t)}(m(t))^j}{j!} \\ &= \mathbb{P}\{\text{Gamma}(n, 1) \geq m(t)\}.\end{aligned}$$

If  $\lim_{t \rightarrow \infty} m(t) = \infty$ , then

$$\mathbb{P}\{S_n = \infty\} = \lim_{t \rightarrow \infty} \mathbb{P}\{S_n > t\} = \mathbb{P}\{\text{Gamma}(n, 1) \geq \lim_{t \rightarrow \infty} m(t)\} = 0$$

and  $S_n$  is a r.v. The density of  $S_n$  is

$$f_{S_n}(t) = e^{-m(t)} \frac{(m(t))^{n-1} m'(t)}{(n-1)!} = e^{-m(t)} \frac{(m(t))^{n-1} \lambda(t)}{(n-1)!}, t \geq 0.$$

If  $\lim_{t \rightarrow \infty} m(t) < \infty$ , then  $S_n$  is a mixed r.v. with

$$\mathbb{P}\{S_n = \infty\} = \mathbb{P}\{\text{Gamma}(n, 1) \geq \lim_{t \rightarrow \infty} m(t)\} > 0$$

and density of its continuous part  $f_{S_n}(t) = \frac{e^{-m(t)}(m(t))^{n-1} \lambda(t)}{(n-1)!}$ ,  $t \geq 0$ .

Let  $T_n = S_n - S_{n-1}$  be the  $n$ -th interarrival time. For a non-homogeneous Poisson process  $\{T_n\}_{n=1}^{\infty}$  are not necessarily independent r.v.'s. For  $0 \leq s \leq t$ ,

$$\begin{aligned}\mathbb{P}\{T_{n+1} > t | S_n = s\} &= \mathbb{P}\{S_{n+1} > s + t | S_n = t\} \\ &= \mathbb{P}\{N(s+t) = n+1 | S_n = t\} = \mathbb{P}\{N(s+t) - N(s) = 1 | S_n = t\} \\ &= \mathbb{P}\{N(s+t) - N(s) = 1\} = e^{-(m(s+t)-m(s))}.\end{aligned}$$

Notice  $S_n = t$  depends on the non-homogeneous Poisson process until time  $t$ . Hence,  $\{N(s+t) - N(s) = 1\}$  and  $\{S_n = t\}$  are independent.

## Example 2

For a non-homogenous Poisson process, the intensity function is given by

$$\lambda(t) = \begin{cases} t & \text{for } 0 < t \leq 4, \\ 4 & \text{for } 10 < t. \end{cases}$$

If  $S_5 = 2$ , calculate the probability that  $S_6 > 5$ .

## Example 2

For a non-homogenous Poisson process, the intensity function is given by

$$\lambda(t) = \begin{cases} t & \text{for } 0 < t \leq 4, \\ 4 & \text{for } 10 < t. \end{cases}$$

If  $S_5 = 2$ , calculate the probability that  $S_6 > 5$ .

**Solution:** We have that

$$\begin{aligned} \mathbb{P}\{S_6 > 5 | S_5 = 2\} &= \mathbb{P}\{N(5) = 5 | S_5 = 2\} \\ &= \mathbb{P}\{N(5) - N(2) = 0 | S_5 = 2\} = P\{N(5) - N(2) = 0\} \end{aligned}$$

and

$$m(5) - m(2) = \int_2^5 \lambda(t) dt = \int_2^4 t dt + \int_4^5 4 dt = 10.$$

Hence,  $\mathbb{P}\{S_6 > 5 | S_5 = 2\} = P\{N(5) - N(2) = 0\} = e^{-10}$ .