Manual for SOA Exam MLC.

Chapter 2. Survival models.

Section 2.1. Survival models.

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Review of Probability theory

Definition 1

Given a set Ω*, a* **probability** $\mathbb{P}$ *on* Ω *is a function defined in the collection of all (subsets) events of* Ω *such that*
(i) $\mathbb{P}(\emptyset) = 0$*.*
(ii) $\mathbb{P}(\Omega) = 1$*.*
(iii) If $\{A_n\}_{n=1}^{\infty}$ *are disjoint events, then*
$\mathbb{P}\{\cup_{n=1}^{\infty} A_n\} = \sum_{n=1}^{\infty} \mathbb{P}\{A_n\}$*.*

Ω is called the **sample space**.

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Ω is called the **sample space**.

Definition 2

A **random variable** X *is function from the sample space Ω into $\mathbb{R}$.*

We will abbreviate random variable into r.v.

Age–at–death

Many insurance concepts depend on accurate estimation of the life span of a person. It is of interest to study the distribution of lives' lifespan. The life span of a person (or any alive entity) can be modeled as a positive (r.v.) random variable.
To model the lifespan of a live, we use **age–at–death** random variable X.
For inanimate objects, **age–at–failure** is the age of an object at the end of termination.

Cumulative distribution function

Definition 3

The **cumulative distribution function** *of a r.v. X is*
$F_X(x) = P\{X \leq x\}$, $x \in \mathbb{R}$.

Theorem 1

A function $F_X : \mathbb{R} \to \Re$ is the (c.d.f.) cumulative distribution function of a r.v. X if and only if:
(i) F_X is nondecreasing, i.e. for each $x_1 \leq x_2$, $F_X(x_1) \leq F_X(x_2)$.
(ii) F_X is right continuous, i.e. for each $x \in \Re$,
$\lim_{h \to 0+} F_X(x+h) = F_X(x)$.
(iii) $\lim_{x \to -\infty} F_X(x) = 0$.
(iv) $\lim_{x \to \infty} F_X(x) = 1$.

The previous theorem gives the following for positive r.v.'s.

Theorem 2

A function $F_X : \mathbb{R} \to \Re$ is the c.d.f. of a positive r.v. X if and only if:

(i) F_X is nondecreasing, i.e. for each $x_1 \le x_2$, $F_X(x_1) \le F_X(x_2)$.

(ii) F_X is right continuous, i.e. for each $x \in \Re$, $\lim_{h \to 0+} F_X(x+h) = F_X(x)$.

(iii) For each $x \le 0$, $F_X(x) = 0$.

(iv) $\lim_{x \to \infty} F_X(x) = 1$.

Example 1

Determine which of the following function is a legitime cumulative distribution function of an age–at–death r.v.:
(i) $F_X(x) = \frac{x+1}{x+3}$, *for* $x \geq 0$.
(ii) $F_X(x) = \frac{x}{2x+1}$, *for* $x \geq 0$.
(iii) $F_X(x) = \frac{x}{x+1}$, *for* $x \geq 0$.

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Solution: (i) $F_X(x) = \frac{x+1}{x+3}$ is not a legitime c.d.f. of an age–at–death because $F_X(0) = \frac{1}{3} \neq 0$.

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Solution: (i) $F_X(x) = \frac{x+1}{x+3}$ is not a legitime c.d.f. of an age–at–death because $F_X(0) = \frac{1}{3} \neq 0$.
(ii) $F_X(x) = \frac{x+1}{x+3}$ is not a legitime c.d.f. of an age–at–death because $\lim_{x\to\infty} F_X(x) = \frac{1}{2} \neq 1$.

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(ii) $F_X(x) = \frac{x+1}{x+3}$ is not a legitime c.d.f. of an age–at–death because $\lim_{x\to\infty} F_X(x) = \frac{1}{2} \neq 1$.
(iii) $F_X(x) = \frac{x}{x+1}$ is a legitime c.d.f. because it satisfies all properties which a c.d.f. should satisfy.

Discrete r.v.

Definition 4

A r.v. X is called **discrete** *if there is a countable set $C \subset \mathbb{R}$ such that $\mathbb{P}\{X \in C\} = 1$.*

If $\mathbb{P}\{X \in C\} = 1$, where $C = \{x_j\}_{j=1}^{\infty}$, then for any set $A \subset \mathbb{R}$,

$$\mathbb{P}\{X \in A\} = \mathbb{P}\{X \in A \cap C\} = \mathbb{P}\{X \in A \cap \{x_j\}_{j=1}^{\infty}\}$$
$$= \mathbb{P}\{X \in \cup_{j:j\geq 1, x_j \in A}\{x_j\}\} = \sum_{j:j\geq 1, x_j \in A} \mathbb{P}\{X = x_j\}.$$

Definition 5

The probability mass function (or frequency function) of the discrete r.v. X is the function $p:\mathbb{R}\to\mathbb{R}$ defined by

$$p(x)=\mathbb{P}\{X=x\},\ x\in\mathbb{R}.$$

If X is a discrete r.v. with p.m.f. p and $A\subset\mathbb{R}$, then

$$\mathbb{P}\{X\in A\}=\sum_{x:x\in A}\mathbb{P}\{X=x\}=\sum_{x:x\in A}p(x).$$

Theorem 3

Let p be the (p.m.f.) probability mass function of the random variable X. Then,
(i) For each $x\geq 0$, $p(x)\geq 0$.
(ii) $\sum_{x\in\mathbb{R}}p(x)=1$.
If a function $p:\mathbb{R}\to\mathbb{R}$ satisfies conditions (i)–(ii) above, then there are a sample space S, a probability measure $\mathbb{P}$ on S and a r.v. $X:S\to\mathbb{R}$ such that X has p.m.f. p.

Continuous r.v.

Definition 6

A r.v. X is called **continuous** *continuous random variable if there exists a nonnegative function f called a (p.d.f.) probability density function of X such that for each $A \subset \mathbb{R}$,*

$$\mathbb{P}\{X \in A\} = \int_A f(x)\,dx = \int_{\mathbb{R}} f(x) I(x \in A)\,dx.$$

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Theorem 4

A function $f : \Re \to \Re$ is the probability density function of a r.v. X if and only if the following two conditions hold:
(i) For each $x \in \mathbb{R}$, $f(x) \geq 0$.
(ii) $\int_{\mathbb{R}} f(x)\,dx = 1$.

If a r.v. is positive and continuous, then $f_X(x) = 0$, for each $x < 0$. So, we only need to define the p.d.f. of an age–at–death for $x \geq 0$.

Example 2

Determine which of the following function is a probability density function of a age–at–death:

(i) $f_X(x) = \frac{1}{(x+1)^2}$, *for* $x \geq 0$.

(ii) $f_X(x) = \frac{1}{(x+1)^3}$, *for* $x \geq 0$.

(iii) $f_X(x) = (2x-1)e^{-x}$, *for* $x \geq 0$.

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(iii) $f_X(x) = (2x-1)e^{-x}$, *for* $x \geq 0$.

Solution: (i) f_X is a density because for each $x \geq 0$, $\frac{1}{(x+1)^2} \geq 0$, and

$$\int_0^\infty \frac{1}{(x+1)^2} = -\frac{1}{x+1}\Big|_0^\infty = 1.$$

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Determine which of the following function is a probability density function of a age–at–death:

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(ii) f_X is not a density function because

$$\int_0^\infty \frac{1}{(x+1)^3} = -\frac{1}{2(x+1)^2}\Big|_0^\infty = \frac{1}{2} \neq 1.$$

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(ii) f_X is not a density function because

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(iii) f_X is not a density function because $(2x-1)e^{-x} < 0$, for each $0 \leq x < \frac{1}{2}$.

Knowing the density f of a r.v. X, the cumulative distribution function of X is given by

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt, x \in \mathbb{R}.$$

Knowing the c.d.f. of a r.v. X, we can find its density using:

Theorem 5

Suppose that the c.d.f. F of a r.v. X satisfies the following conditions:
(i) F is continuous in $\mathbb{R}$.
(ii) There are $a_1, \dots, a_n \in \mathbb{R}$ such that F is continuously differentiable on each of the intervals
$(-\infty, a_1), (a_1, a_2), \dots, (a_{n-1}, a_n), (a_n, \infty)$.
Then, X has a continuous distribution and the p.d.f. of X is given by $f(x) = F'(x)$, except at $a_1, \dots, a_n$.

Example 3

The cumulative distribution function of the random variable X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < -1, \\ \frac{x+1}{4} & \text{if } -1 \leq x < 0, \\ \frac{3x^2+4}{16} & \text{if } 0 \leq x < 2, \\ 1 & \text{if } 2 \leq x. \end{cases}$$

Find the probability density function of X.

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Find the probability density function of X.

Solution: We check that F is continuous and nondecreasing on $\mathbb{R}$. F' exists and it is continuous at each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$ and $(2, \infty)$. A probability density function of X is

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } \ -1 < x \le 0, \\ \frac{3x}{8} & \text{if } 0 < x < 2, \\ 0 & \text{else.} \end{cases}$$

Mixed r.v.

Definition 7

A r.v. X has a **mixed distribution** *if there is a function f and numbers x_j, p_j, $j \geq 1$, with $p_j > 0$, such that for each $A \subset \mathbb{R}$,*

$$\mathbb{P}\{X \in A\} = \int_A f(x)\,dx + \sum_{j:x_j \in A} p_j.$$

A mixed distribution X has two parts: a continuous part and a discrete part. The function f in the previous definition is the p.d.f. of the continuous part of X. The function $p(x) = \mathbb{P}[X = x]$, $x \in \mathbb{R}$, is the p.m.f. of the discrete part of X.
In order to have a r.v., we must have that f is nonnegative and

$$\int_{\mathbb{R}} f(x)\,dx + \sum_{j=1}^{\infty} p_j = 1.$$

Survival function

Definition 8

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Sometimes we will denote the survival function of a r.v. X by s.

Notice that for each $x \geq 0$, $S_X(x) = 1 - F_X(x)$.

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Sometimes we will denote the survival function of a r.v. X by s.
Notice that for each $x \geq 0$, $S_X(x) = 1 - F_X(x)$.

Theorem 6

A function $S_X : [0, \infty) \to \Re$ *is the survival function of a positive r.v.* X *if and only if the following conditions are satisfied:*
(i) S_X *is nonincreasing.*
(ii) S_X *is right continuous.*
(iii) $S_X(0) = 1$.
(iv) $\lim_{x \to \infty} S_X(x) = 0$.

Theorem 7

If the survival function S_X of a r.v. X is continuous everywhere and continuously differentiable except at finitely points, then X has a continuous distribution and the density of X is $f_X(x) = -S'_X(x)$, whenever the derivative exists.

Example 4

Find the density function for the following survival functions:
(i) $s(x) = (1 + x)e^{-x}$, *for* $x \geq 0$.
(ii)

$$s(x) = \begin{cases} 1 - \frac{x^2}{10{,}000} & \text{for } 0 \leq x \leq 100, \\ 0 & \text{for } 100 < x. \end{cases}$$

(iii) $s(x) = \frac{2}{x+2}$, *for* $x \geq 0$.

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Solution: (i) $f_X(x) = xe^{-x}$, for $x \geq 0$.

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Solution: (i) $f_X(x) = xe^{-x}$, for $x \geq 0$.
(ii)

$$f_X(x) = \begin{cases} \frac{2x}{10{,}000} & \text{for } 0 \leq x \leq 100, \\ 0 & \text{for } 100 < x. \end{cases}$$

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$$f_X(x) = \begin{cases} \frac{2x}{10{,}000} & \text{for } 0 \leq x \leq 100, \\ 0 & \text{for } 100 < x. \end{cases}$$

(iii) $f_X(x) = \frac{2}{(x+2)^2}$, for $x \geq 0$.

Terminal age

Often, we will assume that the individuals do not live more than a certain age. This age ω is called the **terminal age** or **limiting age** of the population. So, $S(t) = 0$, for each $t \geq \omega$.

Example 5

Suppose that the survival function of a person is given by $S_X(x) = \frac{90-x}{90}$, *for* $0 \leq x \leq 90$.
(i) Find the probability that a person dies before reaching 20 years old.
(ii) Find the probability that a person lives more than 60 years.

Example 5

Suppose that the survival function of a person is given by $S_X(x) = \frac{90-x}{90}$, *for* $0 \leq x \leq 90$.
(i) Find the probability that a person dies before reaching 20 years old.
(ii) Find the probability that a person lives more than 60 years.

Solution: (i)

$$\mathbb{P}\{X \leq 20\} = 1 - S_X(20) = 1 - \frac{90-20}{90} = \frac{2}{9}.$$

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Suppose that the survival function of a person is given by $S_X(x) = \frac{90-x}{90}$, for $0 \le x \le 90$.
(i) Find the probability that a person dies before reaching 20 years old.
(ii) Find the probability that a person lives more than 60 years.

Solution: (i)

$$\mathbb{P}\{X \le 20\} = 1 - S_X(20) = 1 - \frac{90-20}{90} = \frac{2}{9}.$$

(ii)

$$\mathbb{P}\{X > 60\} = S_X(60) = \frac{90-60}{90} = \frac{1}{3}.$$

Indicator function

Given a set $A \subseteq \mathbb{R}$, the **indicator function** of A is the function

$$I(A) = I(x \in A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Theorem 8

(using the survival function to find an expectation) Let X be a non–negative r.v. with survival function s. Let $h : [0,\infty) \to [0,\infty)$ be a function. Let $H(x) = \int_0^x h(t)\,dt$. Then,

$$E[H(X)] = \int_0^\infty s(t)h(t)\,dt.$$

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$$E[H(X)] = \int_0^\infty s(t)h(t)\, dt.$$

Proof.

Since $H(x) = \int_0^\infty I(x > t)h(t)\, dt$,

$$E[H(X)] = E\left[\int_0^\infty I(X > t)h(t)\, dt\right] = \int_0^\infty E[I(X > t)]h(t)\, dt$$
$$= \int_0^\infty s(t)h(t)\, dt.$$

□

Recall that if $H(x) = \int_0^x h(t)\,dt$, then

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Corollary 1

Let X be a nonnegative r.v. with survival function s. Then,

$$E[X] = \int_0^\infty s(t)\,dt.$$

Recall that if $H(x) = \int_0^x h(t)\,dt$, then

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Corollary 1

Let X be a nonnegative r.v. with survival function s. Then,

$$E[X] = \int_0^\infty s(t)\,dt.$$

Solution: Let $h(t) = 1$, for each $t \geq 0$. Then, $H(x) = \int_0^x h(t)\,dt = x$, for each $x \geq 0$. By Theorem 8,

$$E[X] = E[H(X)] = \int_0^\infty s(t)h(t)\,dt = \int_0^\infty s(t)\,dt.$$

Example 6

Suppose that the survival function of X *is* $s(x) = e^{-x}(x+1)$, $x \geq 0$.
(i) Find $E[X]$ *using that* $E[X] = \int_0^\infty s(t)\,dt$.
(ii) Find the density of X.
(iii) Find $E[X]$ *using that* $E[X] = \int_0^\infty xf(x)\,dx$.

Example 6

Suppose that the survival function of X is $s(x) = e^{-x}(x+1)$, $x \geq 0$.
(i) Find $E[X]$ using that $E[X] = \int_0^\infty s(t)\,dt$.
(ii) Find the density of X.
(iii) Find $E[X]$ using that $E[X] = \int_0^\infty xf(x)\,dx$.

Solution: (i)

$$E[X] = \int_0^\infty s(t)\,dt = \int_0^\infty e^{-x}(x+1)\,dx = 2.$$

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(i) Find $E[X]$ using that $E[X] = \int_0^\infty s(t)\,dt$.
(ii) Find the density of X.
(iii) Find $E[X]$ using that $E[X] = \int_0^\infty xf(x)\,dx$.

Solution: (i)

$$E[X] = \int_0^\infty s(t)\,dt = \int_0^\infty e^{-x}(x+1)\,dx = 2.$$

(ii) The density of X is

$$f(x) = -s'(x) = -e^{-x}(-1)(x+1) - e^{-x}(1) = e^{-x}x.$$

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Suppose that the survival function of X *is* $s(x) = e^{-x}(x+1)$, $x \geq 0$.
(i) Find $E[X]$ *using that* $E[X] = \int_0^\infty s(t)\,dt$.
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$$f(x) = -s'(x) = -e^{-x}(-1)(x+1) - e^{-x}(1) = e^{-x}x.$$

(iii)

$$E[X] = \int_0^\infty xf(x)\,dx = \int_0^\infty x^2e^{-x}\,dx = 2.$$

Recall that if $H(x) = \int_0^x h(t)\,dt$, then

$$E[H(X)] = \int_0^\infty s(t)h(t)\,dt.$$

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Corollary 2

Let X be a nonnegative r.v. with survival function s. Then,

$$E[X^2] = \int_0^\infty s(t)2t\,dt.$$

Recall that if $H(x) = \int_0^x h(t)\,dt$, then

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Corollary 2

Let X be a nonnegative r.v. with survival function s. Then,

$$E[X^2] = \int_0^\infty s(t)2t\,dt.$$

Solution: Let $h(t) = 2t$, for each $t \geq 0$. Hence, $H(x) = \int_0^x h(t)\,dt = x^2$, for each $x \geq 0$. By Theorem 8,

$$E[X^2] = E[H(X)] = \int_0^\infty s(t)h(t)\,dt = \int_0^\infty s(t)2t\,dt.$$

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Corollary 3

Let X be a nonnegative r.v. with survival function s. Let $p > 0$. Then,

$$E[X^p] = \int_0^\infty s(t)pt^{p-1}\,dt.$$

Recall that if $H(x) = \int_0^x h(t)\,dt$, then

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Corollary 3

Let X be a nonnegative r.v. with survival function s. Let $p > 0$. Then,

$$E[X^p] = \int_0^\infty s(t)pt^{p-1}\,dt.$$

Solution: We take $h(t) = pt^{p-1}$, for each $t \geq 0$. Hence, $H(x) = \int_0^x h(t)\,dt = x^p$, for each $x \geq 0$. By Theorem 8, $E[X^p] = \int_0^\infty s(t)pt^{p-1}\,dt$.

Recall that if $H(x) = \int_0^x h(t)\,dt$, then

$$E[H(X)] = \int_0^\infty s(t)h(t)\,dt.$$

Corollary 4

Let X be a nonnegative r.v. with survival function s. Let $a \geq 0$. Then,

$$E[\min(X, a)] = \int_0^a s(t)\,dt.$$

Recall that if $H(x) = \int_0^x h(t)\,dt$, then

$$E[H(X)] = \int_0^\infty s(t)h(t)\,dt.$$

Corollary 4

Let X be a nonnegative r.v. with survival function s. Let $a \geq 0$. Then,

$$E[\min(X,a)] = \int_0^a s(t)\,dt.$$

Solution: Let $h(t) = I(t \in [0,a])$, for each $t \geq 0$. For $x \geq 0$,

$$H(x) = \int_0^x h(t)\,dt = \int_0^x I(t \in [0,a])\,dt = \int_0^{\min(x,a)} dt = \min(x,a).$$

By Theorem 8,

$$E[\min(X,a)] = E[H(X)] = \int_0^\infty s(t)h(t)\,dt = \int_0^a s(t)\,dt.$$

Example 7

Suppose that the survival function of X *is* $s(x) = e^{-x}(x+1)$, $x \geq 0$.
(i) Find $E[\min(X,10)]$ *using that* $E[\min(X,10)] = \int_0^\infty \min(x,10) f(x)\, dx$.
(ii) Find $E[\min(X,10)]$ *using that* $E[\min(X,10)] = \int_0^{10} s(t)\, dt$.

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(ii) Find $E[\min(X, 10)]$ *using that* $E[\min(X, 10)] = \int_0^{10} s(t)\, dt$.

Solution: (i)

$$\begin{aligned}
&\int_0^\infty \min(x, 10) f(x)\, dx = \int_0^{10} x e^{-x} x\, dx + \int_{10}^\infty 10 e^{-x} x\, dx \\
=&2 \int_0^{10} \frac{x^2}{2} e^{-x}\, dx + \int_{10}^\infty 10 e^{-x} x\, dx \\
=&(-2)e^{-x} \left(\frac{x^2}{2} + x + 1 \right) \Big|_0^{10} - 10 e^{-x}(x+1) \Big|_{10}^\infty \\
=&2 - 2e^{-10}(61) + 10e^{-10}(11) = 2 - 12e^{-10}.
\end{aligned}$$

Example 7

Suppose that the survival function of X is $s(x) = e^{-x}(x+1)$, $x \geq 0$.
(i) Find $E[\min(X,10)]$ using that $E[\min(X,10)] = \int_0^\infty \min(x,10) f(x)\,dx$.
(ii) Find $E[\min(X,10)]$ using that $E[\min(X,10)] = \int_0^{10} s(t)\,dt$.

Solution: (ii)

$$\int_0^{10} s(t)\,dt = \int_0^{10} e^{-t}(t+1)\,dt = \int_0^{10} e^{-t} t dt + \int_0^{10} e^{-t}\,dt$$

$$= -e^{-t}(t+1)\Big|_0^{10} - e^{-t}\Big|_0^{10} = 1 - 11e^{-10} + 1 - e^{-10} = 2 - 12e^{-10}.$$

Theorem 9

Let X be a discrete r.v. whose possible values are nonnegative integers. Let $h : [0, \infty) \to [0, \infty)$ be a function. Let $H(x) = \int_0^x h(t)\,dt$. Then,

$$E[H(X)] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(H(k) - H(k-1)).$$

Proof: We have that $s(t) = \mathbb{P}\{X \geq k\}$, for $k - 1 \leq t < k$. Hence,

$$E[H(X)] = \int_0^{\infty} s(t)h(t)\,dt = \sum_{k=1}^{\infty} \int_{k-1}^{k} s(t)h(t)\,dt$$

$$= \sum_{k=1}^{\infty} \int_{k-1}^{k} \mathbb{P}\{X \geq k\}h(t)\,dt = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} \int_{k-1}^{k} h(t)\,dt$$

$$= \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(H(k) - H(k-1)).$$

Recall that if $H(x) = \int_0^x h(t)\,dt$, then,

$$E[H(X)] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(H(k) - H(k-1)).$$

This implies that

$$E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\},$$

$$E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(k^2 - (k-1)^2) = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(2k-1)$$

and

$$E[\min(X, a)] = \sum_{k=1}^{a} \mathbb{P}\{X \geq k\},$$

where a is positive integer.

Example 8

Let X be a discrete r.v. with probability mass function given by the following table,

k	0	1	2
$\mathbb{P}\{X = k\}$	0.2	0.3	0.5

(i) Find $E[X]$ and $E[X^2]$, using that
$E[H(X)] = \sum_{k=0}^{\infty} H(k)\mathbb{P}\{X = k\}$.
(ii) Find $E[X]$ and $E[X^2]$, using that $E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}$ and
$E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(2k-1)$.

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Solution: (i) We have that

$$E[X] = (0)(0.2) + (1)(0.3) + (2)(0.5) = 1.3$$
$$E[X^2] = (0)^2(0.2) + (1)^2(0.3) + (2)^2(0.5) = 2.3.$$

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(i) Find $E[X]$ and $E[X^2]$, using that
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(ii) Find $E[X]$ and $E[X^2]$, using that $E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}$ and
$E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(2k - 1)$.

Solution: (ii) We have that $\mathbb{P}\{X \geq 1\} = 0.8$, $\mathbb{P}\{X \geq 2\} = 0.5$, and $\mathbb{P}\{X \geq k\} = 0$, for each $k \geq 3$. Hence,

$$E[X] = \mathbb{P}\{X \geq 1\} + \mathbb{P}\{X \geq 2\} = 0.8 + 0.5 = 1.3$$

$$E[X^2] = \mathbb{P}\{X \geq 1\}((2)(1) - 1) + \mathbb{P}\{X \geq 2\}((2)(2) - 1)$$

$$= 0.8 + 0.5(3) = 2.3.$$

Definition 9

Given $0 < p < 1$, *the* $100p$**–th percentile** *(or* p**–th quantile***) of a r.v.* X *is a value such that*

$$\mathbb{P}\{X < \xi_p\} \le p \le \mathbb{P}\{X \le \xi_p\}.$$

Usually $\mathbb{P}\{X \le \xi_p\} = p$. If X has a continuous distribution, then $\mathbb{P}\{X < \xi_p\} = \mathbb{P}\{X \le \xi_p\}$ and $\mathbb{P}\{X \le \xi_p\} = p$.

Theorem 10

If X has a uniform distribution on the interval (a, b), then the p–th quantile ξ_p of X is $a + (b - a)p$.

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Proof: We have that

$$p = \mathbb{P}\{X \leq \xi_p\} = \int_a^{\xi_p} \frac{1}{b-a} + \, dt = \frac{\xi_p - a}{b - a}.$$

So, $\xi_p = a + (b - a)p$.

Definition 10

A **median** *m of a r.v. X is a value such that*
$\mathbb{P}\{X < m\} \le \frac{1}{2} \le \mathbb{P}\{X \le m\}$.

Definition 11

The **first quartile** *Q_1 of a r.v. X is the 25–th percentile of the r.v. X. The* **third quartile** *Q_3 of a r.v. X is the 75–th percentile of the r.v. X.*

Usually, the range of a r.v. X is divided in four parts with probability 0.25 each by the numbers $-\infty, Q_1, m, Q_3, \infty$.

Interval	$(-\infty, Q_1)$	(Q_1, m)	(m, Q_3)	(Q_3, ∞)
Probability	25 %	25%	25%	25%

Example 9

Suppose that the age–at–failure r.v. X has density

$$f_X(x) = \begin{cases} \frac{5x^4}{k^5} & \text{if } \; 0 < x < k, \\ 0 & \text{else.} \end{cases}$$

Suppose that the expected age–at–failure is 70 years. Find the median age–at–failure.

Example 9

Suppose that the age–at–failure r.v. X has density

$$f_X(x) = \begin{cases} \frac{5x^4}{k^5} & \text{if } 0 < x < k, \\ 0 & \text{else.} \end{cases}$$

Suppose that the expected age–at–failure is 70 years. Find the median age–at–failure.

Solution: Since

$$70 = E[X] = \int_0^k x \frac{5x^4}{k^5}\, dx = \frac{5x^6}{6k^5} \Big|_0^k = \frac{5k}{6},$$

$k = \frac{(70)(6)}{5} = 84$. Let m be the median age–at–failure. Then,

$$\frac{1}{2} = \int_0^m \frac{5x^4}{(84)^5}\, dx = \frac{x^5}{(84)^5} \Big|_0^m = \frac{m^5}{(84)^5},$$

and $m = \frac{84}{2^{\frac{1}{5}}} = 73.12624732.$

Theorem 11

Let X be a continuous r.v. with density function f_X. Let $0 < p < 1$. Suppose that there are $-\infty \le a < b \le \infty$ such that:
(i) $f_X(x) = 0$, if $x \notin (a, b)$.
(ii) $f_X(x)$ is continuous and positive in (a, b).
Then, there exists ξ_p such that $F_X(\xi_p) = p$. Moreover, ξ_p is unique.

Theorem 12

Let X be a r.v. with range (a,b) and density f_X. Let $0 < p < 1$. Let $h:(a,b)\to(c,d)$ be a one–to–one onto function.
(i) Let ξ_p be a p–th quantile of X such that $\mathbb{P}\{X < \xi_p\} = p = \mathbb{P}\{X \le \xi_p\}$. If h is nonincreasing, then a p–the quantile of Y is $\zeta_p = h(\xi_p)$.
(ii) Let ξ_{1-p} be a $(1-p)$–th quantile of X such that $\mathbb{P}\{X < \xi_{1-p}\} = 1-p = \mathbb{P}\{X \le \xi_{1-p}\}$. If h is nondecreasing, then a p–the quantile of Y is $\zeta_p = h(\xi_{1-p})$.

Example 10

Suppose that the age–at–failure r.v. X has density

$$f_X(x) = \begin{cases} \frac{5x^4}{(84)^5} & \text{if } 0 < x < 84, \\ 0 & \text{else.} \end{cases}$$

Find the three quartiles of $(1000)(1.06)^{-X}$.

Example 10

Suppose that the age–at–failure r.v. X has density

$$f_X(x) = \begin{cases} \frac{5x^4}{(84)^5} & \text{if } \ 0 < x < 84, \\ 0 & \text{else.} \end{cases}$$

Find the three quartiles of $(1000)(1.06)^{-X}$.

Solution: (i) let $h(x) = (1000)(1.06)^{-x}$, $x \geq 0$. h is a decreasing function. Let ξ_p be p–th quantile of the r.v. X. Let ζ_p be p–th quantile of the r.v. $h(X)$. By the previous theorem, $\zeta_p = h(\xi_{1-p})$. Hence,

$$\zeta_{0.25} = h(\xi_{0.75}) = (1000)(1.06)^{-79.30335095} = 9.843738901,$$
$$\zeta_{0.5} = h(\xi_{0.5}) = (1000)(1.06)^{-73.12624732} = 14.10837641,$$
$$\zeta_{0.75} = h(\xi_{0.25}) = (1000)(1.06)^{-63.66009579} = 24.49210954,$$

