Chapter 2. Survival models.

Section 2.1. Survival models.

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Extract from:
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Review of Probability theory

**Definition 1**

*Given a set $\Omega$, a **probability** $\mathbb{P}$ on $\Omega$ is a function defined in the collection of all (subsets) events of $\Omega$ such that*

(i) $\mathbb{P}(\emptyset) = 0$.

(ii) $\mathbb{P}(\Omega) = 1$.

(iii) *If $\{A_n\}_{n=1}^{\infty}$ are disjoint events, then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mathbb{P}\{A_n\}.$$  

$\Omega$ is called the **sample space**.
Review of Probability theory

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$$\mathbb{P}\{\bigcup_{n=1}^{\infty} A_n\} = \sum_{n=1}^{\infty} \mathbb{P}\{A_n\}.$$ 

$\Omega$ is called the **sample space**.

Definition 2

*A **random variable** $X$ is function from the sample space $\Omega$ into $\mathbb{R}$. We will abbreviate random variable into r.v.*
Age–at–death

Many insurance concepts depend on accurate estimation of the life span of a person. It is of interest to study the distribution of lives’ lifespan. The life span of a person (or any alive entity) can be modeled as a positive (r.v.) random variable.
To model the lifespan of a live, we use age–at–death random variable $X$.
For inanimate objects, age–at–failure is the age of an object at the end of termination.
Chapter 2. Survival models.

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Cumulative distribution function

Definition 3
The cumulative distribution function of a r.v. $X$ is $F_X(x) = P\{X \leq x\}$, $x \in \mathbb{R}$.

Theorem 1
A function $F_X : \mathbb{R} \to \mathbb{R}$ is the (c.d.f.) cumulative distribution function of a r.v. $X$ if and only if:

(i) $F_X$ is nondecreasing, i.e. for each $x_1 \leq x_2$, $F_X(x_1) \leq F_X(x_2)$.

(ii) $F_X$ is right continuous, i.e. for each $x \in \mathbb{R}$,

$$\lim_{h \to 0^+} F_X(x + h) = F_X(x).$$

(iii) $\lim_{x \to -\infty} F_X(x) = 0$.

(iv) $\lim_{x \to \infty} F_X(x) = 1$. 

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The previous theorem gives the following for positive r.v.’s.

**Theorem 2**

A function $F_X : \mathbb{R} \rightarrow \mathbb{R}$ is the c.d.f. of a positive r.v. $X$ if and only if:

(i) $F_X$ is nondecreasing, i.e. for each $x_1 \leq x_2$, $F_X(x_1) \leq F_X(x_2)$.

(ii) $F_X$ is right continuous, i.e. for each $x \in \mathbb{R}$,

$$\lim_{h \rightarrow 0^+} F_X(x + h) = F_X(x).$$

(iii) For each $x \leq 0$, $F_X(x) = 0$.

(iv) $\lim_{x \rightarrow \infty} F_X(x) = 1$. 
Example 1

Determine which of the following function is a legitimate cumulative distribution function of an age–at–death r.v.:

(i) \( F_X(x) = \frac{x+1}{x+3} \), for \( x \geq 0 \).
(ii) \( F_X(x) = \frac{x}{2x+1} \), for \( x \geq 0 \).
(iii) \( F_X(x) = \frac{x}{x+1} \), for \( x \geq 0 \).
Example 1

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(i) \( F_X(x) = \frac{x+1}{x+3}, \) for \( x \geq 0. \)
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(iii) \( F_X(x) = \frac{x}{x+1}, \) for \( x \geq 0. \)

Solution: (i) \( F_X(x) = \frac{x+1}{x+3} \) is not a legitimate c.d.f. of an age–at–death because \( F_X(0) = \frac{1}{3} \neq 0. \)
Example 1

Determine which of the following function is a legitimate cumulative distribution function of an age–at–death r.v.:

(i) \( F_X(x) = \frac{x+1}{x+3} \), for \( x \geq 0 \).

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Solution: (i) \( F_X(x) = \frac{x+1}{x+3} \) is not a legitimate c.d.f. of an age–at–death because \( F_X(0) = \frac{1}{3} \neq 0 \).

(ii) \( F_X(x) = \frac{x+1}{x+3} \) is not a legitimate c.d.f. of an age–at–death because \( \lim_{x \to \infty} F_X(x) = \frac{1}{2} \neq 1 \).
Example 1

Determine which of the following function is a legitime cumulative distribution function of an age–at–death r.v.:

(i) \( F_X(x) = \frac{x+1}{x+3} \), for \( x \geq 0 \).
(ii) \( F_X(x) = \frac{x}{2x+1} \), for \( x \geq 0 \).
(iii) \( F_X(x) = \frac{x}{x+1} \), for \( x \geq 0 \).

Solution: (i) \( F_X(x) = \frac{x+1}{x+3} \) is not a legitime c.d.f. of an age–at–death because \( F_X(0) = \frac{1}{3} \neq 0 \).
(ii) \( F_X(x) = \frac{x+1}{x+3} \) is not a legitime c.d.f. of an age–at–death because \( \lim_{x \to \infty} F_X(x) = \frac{1}{2} \neq 1 \).
(iii) \( F_X(x) = \frac{x}{x+1} \) is a legitime c.d.f. because it satisfies all properties which a c.d.f. should satisfy.
Discrete r.v.

Definition 4

A r.v. $X$ is called **discrete** if there is a countable set $C \subset \mathbb{R}$ such that $P\{X \in C\} = 1$.

If $P\{X \in C\} = 1$, where $C = \{x_j\}_{j=1}^\infty$, then for any set $A \subset \mathbb{R}$,

$$P\{X \in A\} = P\{X \in A \cap C\} = P\{X \in A \cap \{x_j\}_{j=1}^\infty\}$$

$$= P\{X \in \bigcup_{j:j \geq 1,x_j \in A}\{x_j\}\} = \sum_{j:j \geq 1,x_j \in A} P\{X = x_j\}.$$
Definition 5
The probability mass function (or frequency function) of the discrete r.v. $X$ is the function $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(x) = \mathbb{P}\{X = x\}, \ x \in \mathbb{R}.$$ 

If $X$ is a discrete r.v. with p.m.f. $p$ and $A \subset \mathbb{R}$, then

$$\mathbb{P}\{X \in A\} = \sum_{x : x \in A} \mathbb{P}\{X = x\} = \sum_{x : x \in A} p(x).$$

Theorem 3
Let $p$ be the (p.m.f.) probability mass function of the random variable $X$. Then,

(i) For each $x \geq 0$, $p(x) \geq 0$.

(ii) $\sum_{x \in \mathbb{R}} p(x) = 1$.

If a function $p : \mathbb{R} \to \mathbb{R}$ satisfies conditions (i)–(ii) above, then there are a sample space $S$, a probability measure $\mathbb{P}$ on $S$ and a r.v. $X : S \to \mathbb{R}$ such that $X$ has p.m.f. $p$. 
Definition 6
A r.v. $X$ is called \textit{continuous} if there exists a nonnegative function $f$ called a (p.d.f.) probability density function of $X$ such that for each $A \subset \mathbb{R}$,

$$
\mathbb{P}\{X \in A\} = \int_A f(x) \, dx = \int_{\mathbb{R}} f(x) I(x \in A) \, dx.
$$

Definition 6
A r.v. $X$ is called **continuous** continuous random variable if there exists a nonnegative function $f$ called a (p.d.f.) probability density function of $X$ such that for each $A \subset \mathbb{R}$,

$$P\{X \in A\} = \int_A f(x) \, dx = \int_{\mathbb{R}} f(x) I(x \in A) \, dx.$$ 

Theorem 4
A function $f : \mathbb{R} \to \mathbb{R}$ is the probability density function of a r.v. $X$ if and only if the following two conditions hold:
1. For each $x \in \mathbb{R}$, $f(x) \geq 0$.
2. $\int_{\mathbb{R}} f(x) \, dx = 1$. 
If a r.v. is positive and continuous, then $f_X(x) = 0$, for each $x < 0$. So, we only need to define the p.d.f. of an age–at–death for $x \geq 0$. 
Example 2

Determine which of the following function is a probability density function of a age–at–death:

(i) \( f_X(x) = \frac{1}{(x+1)^2} \), for \( x \geq 0 \).

(ii) \( f_X(x) = \frac{1}{(x+1)^3} \), for \( x \geq 0 \).

(iii) \( f_X(x) = (2x - 1)e^{-x} \), for \( x \geq 0 \).
Example 2

Determine which of the following function is a probability density function of a age–at–death:

(i) \( f_X(x) = \frac{1}{(x+1)^2} \), for \( x \geq 0 \).

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(iii) \( f_X(x) = (2x - 1)e^{-x} \), for \( x \geq 0 \).

**Solution:** (i) \( f_X \) is a density because for each \( x \geq 0 \), \( \frac{1}{(x+1)^2} \geq 0 \), and

\[
\int_0^\infty \frac{1}{(x+1)^2} = - \left. \frac{1}{x+1} \right|_0^\infty = 1.
\]
Example 2

Determine which of the following function is a probability density function of an age–at–death:

(i) \( f_X(x) = \frac{1}{(x+1)^2}, \text{ for } x \geq 0. \)

(ii) \( f_X(x) = \frac{1}{(x+1)^3}, \text{ for } x \geq 0. \)

(iii) \( f_X(x) = (2x - 1)e^{-x}, \text{ for } x \geq 0. \)

Solution: (i) \( f_X \) is a density because for each \( x \geq 0 \), \( \frac{1}{(x+1)^2} \geq 0 \), and

\[
\int_0^\infty \frac{1}{(x+1)^2} = -\left. \frac{1}{x+1} \right|_0^\infty = 1.
\]

(ii) \( f_X \) is not a density function because

\[
\int_0^\infty \frac{1}{(x+1)^3} = -\left. \frac{1}{2(x+1)^2} \right|_0^\infty = \frac{1}{2} \neq 1.
\]
Example 2

Determine which of the following function is a probability density function of a age-at-death:

(i) \( f_X(x) = \frac{1}{(x+1)^2} , \text{ for } x \geq 0. \)

(ii) \( f_X(x) = \frac{1}{(x+1)^3} , \text{ for } x \geq 0. \)

(iii) \( f_X(x) = (2x - 1)e^{-x} , \text{ for } x \geq 0. \)

**Solution:**

(i) \( f_X \) is a density because for each \( x \geq 0, \frac{1}{(x+1)^2} \geq 0, \) and

\[
\int_0^\infty \frac{1}{(x+1)^2} = -\frac{1}{x+1} \bigg|_0^\infty = 1.
\]

(ii) \( f_X \) is not a density function because

\[
\int_0^\infty \frac{1}{(x+1)^3} = -\frac{1}{2(x+1)^2} \bigg|_0^\infty = \frac{1}{2} \neq 1.
\]

(iii) \( f_X \) is not a density function because \((2x - 1)e^{-x} < 0, \text{ for each } 0 \leq x < \frac{1}{2}.\)
Knowing the density $f$ of a r.v. $X$, the cumulative distribution function of $X$ is given by

$$F_X(x) = \int_{-\infty}^{x} f(t) \, dt, \quad x \in \mathbb{R}.$$ 

Knowing the c.d.f. of a r.v. $X$, we can find its density using:

**Theorem 5**

*Suppose that the c.d.f. $F$ of a r.v. $X$ satisfies the following conditions:*

(i) $F$ is continuous in $\mathbb{R}$.

(ii) There are $a_1, \ldots, a_n \in \mathbb{R}$ such that $F$ is continuously differentiable on each of the intervals

$(-\infty, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, \infty)$.

*Then, $X$ has a continuous distribution and the p.d.f. of $X$ is given by $f(x) = F'(x)$, except at $a_1, \ldots, a_n$.***
Example 3

The cumulative distribution function of the random variable $X$ is given by

$$F(x) = \begin{cases} 
0 & \text{if } x < -1, \\
\frac{x+1}{4} & \text{if } -1 \leq x < 0, \\
\frac{3x^2+4}{16} & \text{if } 0 \leq x < 2, \\
1 & \text{if } 2 \leq x.
\end{cases}$$

Find the probability density function of $X$. 
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1 & \text{if } 2 \leq x.
\end{cases}
\]

Find the probability density function of $X$.

Solution: We check that $F$ is continuous and nondecreasing on $\mathbb{R}$. $F'$ exists and it is continuous at each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$ and $(2, \infty)$. A probability density function of $X$ is

\[
f(x) = \begin{cases} 
\frac{1}{4} & \text{if } -1 < x \leq 0, \\
\frac{3x}{8} & \text{if } 0 < x < 2, \\
0 & \text{else}.
\end{cases}
\]
Definition 7

A r.v. $X$ has a **mixed distribution** if there is a function $f$ and numbers $x_j, p_j, j \geq 1$, with $p_j > 0$, such that for each $A \subseteq \mathbb{R}$,

$$
P\{X \in A\} = \int_A f(x) \, dx + \sum_{j : x_j \in A} p_j.
$$

A mixed distribution $X$ has two parts: a continuous part and a discrete part. The function $f$ in the previous definition is the p.d.f. of the continuous part of $X$. The function $p(x) = P[X = x]$, $x \in \mathbb{R}$, is the p.m.f. of the discrete part of $X$.

In order to have a r.v., we must have that $f$ is nonnegative and

$$
\int_{\mathbb{R}} f(x) \, dx + \sum_{j=1}^{\infty} p_j = 1.
$$
Definition 8

The survival function of a r.v. $X$ is the function $S_X(x) = P\{X > x\}, x \in \mathbb{R}$.
**Survival function**

**Definition 8**

The **survival function** of a r.v. $X$ is the function

$$S_X(x) = \mathbb{P}\{X > x\}, \; x \in \mathbb{R}.$$  

Sometimes we will denote the survival function of a r.v. $X$ by $s$. Notice that for each $x \geq 0$, $S_X(x) = 1 - F_X(x)$. 

Survival function

Definition 8
The survival function of a r.v. $X$ is the function $S_X(x) = \mathbb{P}\{X > x\}$, $x \in \mathbb{R}$.

Sometimes we will denote the survival function of a r.v. $X$ by $s$. Notice that for each $x \geq 0$, $S_X(x) = 1 - F_X(x)$.

Theorem 6
A function $S_X : [0, \infty) \rightarrow \mathbb{R}$ is the survival function of a positive r.v. $X$ if and only if the following conditions are satisfied:

(i) $S_X$ is nonincreasing.
(ii) $S_X$ is right continuous.
(iii) $S_X(0) = 1$.
(iv) $\lim_{x \to \infty} S_X(x) = 0$. 
Theorem 7

If the survival function $S_X$ of a r.v. $X$ is continuous everywhere and continuously differentiable except at finitely points, then $X$ has a continuous distribution and the density of $X$ is $f_X(x) = -S_X'(x)$, whenever the derivative exists.
Example 4

*Find the density function for the following survival functions:*

(i) \( s(x) = (1 + x)e^{-x}, \) for \( x \geq 0. \)

(ii) \[
s(x) = \begin{cases} 
1 - \frac{x^2}{10,000} & \text{for } 0 \leq x \leq 100, \\
0 & \text{for } 100 < x.
\end{cases}
\]

(iii) \( s(x) = \frac{2}{x+2}, \) for \( x \geq 0. \)
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**Solution:** (i) \( f_X(x) = xe^{-x}, \text{ for } x \geq 0. \)
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    \end{cases}
\]

(iii) \( s(x) = \frac{2}{x+2}, \) for \( x \geq 0. \)

Solution: (i) \( f_X(x) = xe^{-x}, \) for \( x \geq 0. \)

(ii) \[
    f_X(x) = \begin{cases} 
    \frac{2x}{10,000} & \text{for } 0 \leq x \leq 100, \\
    0 & \text{for } 100 < x.
    \end{cases}
\]
Example 4

Find the density function for the following survival functions:

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    s(x) = \begin{cases} 
    1 - \frac{x^2}{10,000} & \text{for } 0 \leq x \leq 100, \\
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    \end{cases}
\]

(iii) \( s(x) = \frac{2}{x+2}, \) for \( x \geq 0. \)

Solution: (i) \( f_X(x) = xe^{-x}, \) for \( x \geq 0. \)

(ii) \[
    f_X(x) = \begin{cases} 
    \frac{2x}{10,000} & \text{for } 0 \leq x \leq 100, \\
    0 & \text{for } 100 < x. 
    \end{cases}
\]

(iii) \( f_X(x) = \frac{2}{(x+2)^2}, \) for \( x \geq 0. \)
Terminal age

Often, we will assume that the individuals do not live more than a certain age. This age $\omega$ is called the **terminal age** or **limiting age** of the population. So, $S(t) = 0$, for each $t \geq \omega$. 
Example 5

Suppose that the survival function of a person is given by

\[ S_X(x) = \frac{90-x}{90}, \text{ for } 0 \leq x \leq 90. \]

(i) Find the probability that a person dies before reaching 20 years old.

(ii) Find the probability that a person lives more than 60 years old.
Example 5

Suppose that the survival function of a person is given by
\[ S_X(x) = \frac{90 - x}{90}, \text{ for } 0 \leq x \leq 90. \]

(i) Find the probability that a person dies before reaching 20 years old.

(ii) Find the probability that a person lives more than 60 years.

Solution: (i)

\[ \mathbb{P}\{X \leq 20\} = 1 - S_X(20) = 1 - \frac{90 - 20}{90} = \frac{2}{9}. \]
Example 5

Suppose that the survival function of a person is given by 

\[ S_X(x) = \frac{90-x}{90} \], for \( 0 \leq x \leq 90. \)

(i) Find the probability that a person dies before reaching 20 years old.

(ii) Find the probability that a person lives more than 60 years.

Solution: (i)

\[
\mathbb{P}\{X \leq 20\} = 1 - S_X(20) = 1 - \frac{90 - 20}{90} = \frac{2}{9}.
\]

(ii)

\[
\mathbb{P}\{X > 60\} = S_X(60) = \frac{90 - 60}{90} = \frac{1}{3}.
\]
Given a set $A \subseteq \mathbb{R}$, the **indicator function** of $A$ is the function

$$I(A) = I(\{x \in A\}) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$
Theorem 8
(using the survival function to find an expectation) Let $X$ be a non-negative r.v. with survival function $s$. Let $h : [0, \infty) \to [0, \infty)$ be a function. Let $H(x) = \int_0^x h(t) \, dt$. Then,

$$E[H(X)] = \int_0^\infty s(t)h(t) \, dt.$$
Theorem 8
(Using the survival function to find an expectation) Let $X$ be a non-negative r.v. with survival function $s$. Let $h : [0, \infty) \rightarrow [0, \infty)$ be a function. Let $H(x) = \int_0^x h(t) dt$. Then,

$$E[H(X)] = \int_0^\infty s(t)h(t) dt.$$

Proof.
Since $H(x) = \int_0^\infty I(x > t)h(t) dt$,

$$E[H(X)] = E \left[ \int_0^\infty I(X > t)h(t) dt \right] = \int_0^\infty E[I(X > t)]h(t) dt$$

$$= \int_0^\infty s(t)h(t) dt.$$

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Recall that if $H(x) = \int_0^x h(t) \, dt$, then

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$$E[H(X)] = \int_0^\infty s(t)h(t) \, dt.$$ 

**Corollary 1**

*Let $X$ be a nonnegative r.v. with survival function $s$. Then,*

$$E[X] = \int_0^\infty s(t) \, dt.$$
Recall that if $H(x) = \int_0^x h(t) \, dt$, then

$$E[H(X)] = \int_0^\infty s(t)h(t) \, dt.$$ 

**Corollary 1**

Let $X$ be a nonnegative r.v. with survival function $s$. Then,

$$E[X] = \int_0^\infty s(t) \, dt.$$ 

**Solution:** Let $h(t) = 1$, for each $t \geq 0$. Then,

$H(x) = \int_0^x h(t) \, dt = x$, for each $x \geq 0$. By Theorem 8,

$$E[X] = E[H(X)] = \int_0^\infty s(t)h(t) \, dt = \int_0^\infty s(t) \, dt.$$
Example 6

Suppose that the survival function of $X$ is $s(x) = e^{-x}(x + 1)$, $x \geq 0$.

(i) Find $E[X]$ using that $E[X] = \int_0^\infty s(t) \, dt$.

(ii) Find the density of $X$.

(iii) Find $E[X]$ using that $E[X] = \int_0^\infty x f(x) \, dx$. 
Example 6

Suppose that the survival function of $X$ is $s(x) = e^{-x}(x + 1)$, $x \geq 0$.

(i) Find $E[X]$ using that $E[X] = \int_0^\infty s(t) \, dt$.

(ii) Find the density of $X$.

(iii) Find $E[X]$ using that $E[X] = \int_0^\infty x f(x) \, dx$.

Solution: (i)

$$E[X] = \int_0^\infty s(t) \, dt = \int_0^\infty e^{-x}(x + 1) \, dx = 2.$$
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(ii) Find the density of $X$.
(iii) Find $E[X]$ using that $E[X] = \int_0^\infty x f(x) \, dx$.

Solution:  

(i) 

$$E[X] = \int_0^\infty s(t) \, dt = \int_0^\infty e^{-x}(x + 1) \, dx = 2.$$ 

(ii) The density of $X$ is 

$$f(x) = -s'(x) = -e^{-x}(-1)(x + 1) - e^{-x}(1) = e^{-x}x.$$
Example 6

Suppose that the survival function of $X$ is $s(x) = e^{-x}(x + 1)$, $x \geq 0$.

(i) Find $E[X]$ using that $E[X] = \int_0^\infty s(t) \, dt$.

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(iii) Find $E[X]$ using that $E[X] = \int_0^\infty x f(x) \, dx$.

Solution: (i)

$$E[X] = \int_0^\infty s(t) \, dt = \int_0^\infty e^{-x}(x + 1) \, dx = 2.$$

(ii) The density of $X$ is

$$f(x) = -s'(x) = -e^{-x}(-1)(x + 1) - e^{-x}(1) = e^{-x}x.$$

(iii)

$$E[X] = \int_0^\infty x f(x) \, dx = \int_0^\infty x^2e^{-x} \, dx = 2.$$
Recall that if $H(x) = \int_0^x h(t) \, dt$, then

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Recall that if $H(x) = \int_0^x h(t) \, dt$, then

$$E[H(X)] = \int_0^\infty s(t)h(t) \, dt.$$ 

**Corollary 2**

Let $X$ be a nonnegative r.v. with survival function $s$. Then,

$$E[X^2] = \int_0^\infty s(t)2t \, dt.$$
Recall that if $H(x) = \int_0^x h(t) \, dt$, then

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**Corollary 2**

Let $X$ be a nonnegative r.v. with survival function $s$. Then,

$$E[X^2] = \int_0^\infty s(t)2t \, dt.$$ 

**Solution:** Let $h(t) = 2t$, for each $t \geq 0$. Hence, $H(x) = \int_0^x h(t) \, dt = x^2$, for each $x \geq 0$. By Theorem 8,

$$E[X^2] = E[H(X)] = \int_0^\infty s(t)h(t) \, dt = \int_0^\infty s(t)2t \, dt.$$
Recall that if \( H(x) = \int_0^x h(t) \, dt \), then

\[
E[H(X)] = \int_0^\infty s(t) h(t) \, dt.
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**Corollary 3**

*Let $X$ be a nonnegative r.v. with survival function $s$. Let $p > 0$. Then,

$$E[X^p] = \int_0^\infty s(t)pt^{p-1} \, dt.$$*
Recall that if \( H(x) = \int_0^x h(t) \, dt \), then

\[
E[H(X)] = \int_0^\infty s(t)h(t) \, dt.
\]

**Corollary 3**

Let \( X \) be a nonnegative r.v. with survival function \( s \). Let \( p > 0 \). Then,

\[
E[X^p] = \int_0^\infty s(t)pt^{p-1} \, dt.
\]

**Solution:** We take \( h(t) = pt^{p-1} \), for each \( t \geq 0 \). Hence, \( H(x) = \int_0^x h(t) \, dt = x^p \), for each \( x \geq 0 \). By Theorem 8, \( E[X^p] = \int_0^\infty s(t)pt^{p-1} \, dt \).
Recall that if \( H(x) = \int_0^x h(t) \, dt \), then
\[
E[H(X)] = \int_0^\infty s(t) h(t) \, dt.
\]

**Corollary 4**

*Let \( X \) be a nonnegative r.v. with survival function \( s \). Let \( a \geq 0 \). Then,
\[
E[\min(X, a)] = \int_0^a s(t) \, dt.
\]
Recall that if $H(x) = \int_0^x h(t) \, dt$, then

$$E[H(X)] = \int_0^{\infty} s(t) h(t) \, dt.$$ 

**Corollary 4**

*Let $X$ be a nonnegative r.v. with survival function $s$. Let $a \geq 0$. Then,*

$$E[\min(X, a)] = \int_0^a s(t) \, dt.$$ 

**Solution:** Let $h(t) = I(t \in [0, a])$, for each $t \geq 0$. For $x \geq 0$,

$$H(x) = \int_0^x h(t) \, dt = \int_0^x I(t \in [0, a]) \, dt = \int_0^{\min(x, a)} \, dt = \min(x, a).$$

By Theorem 8,

$$E[\min(X, a)] = E[H(X)] = \int_0^{\infty} s(t) h(t) \, dt = \int_0^a s(t) \, dt.$$
Example 7

Suppose that the survival function of $X$ is $s(x) = e^{-x}(x + 1)$, $x \geq 0$.

(i) Find $E[\min(X, 10)]$ using that

$E[\min(X, 10)] = \int_{0}^{\infty} \min(x, 10)f(x)\,dx$.

(ii) Find $E[\min(X, 10)]$ using that $E[\min(X, 10)] = \int_{0}^{10} s(t)\,dt$. 

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Solution: (i)

$$\int_0^\infty \min(x, 10)f(x)\,dx = \int_0^{10} xe^{-x} \,dx + \int_{10}^\infty 10e^{-x} \,dx$$

$$= 2 \int_0^{10} \frac{x^2}{2} e^{-x} \,dx + \int_{10}^\infty 10e^{-x} \,dx$$

$$= (-2)e^{-x} \left( \frac{x^2}{2} + x + 1 \right) \bigg|_0^{10} - 10e^{-x}(x + 1) \bigg|_{10}^\infty$$

$$= 2 - 2e^{-10}(61) + 10e^{-10}(11) = 2 - 12e^{-10}.$$
Example 7

Suppose that the survival function of $X$ is $s(x) = e^{-x}(x + 1)$, $x \geq 0$.

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$(ii)$ Find $E[\min(X, 10)]$ using that $E[\min(X, 10)] = \int_{0}^{10} s(t) \, dt$.

**Solution:** (ii)

$$
\int_{0}^{10} s(t) \, dt = \int_{0}^{10} e^{-t}(t + 1) \, dt = \int_{0}^{10} e^{-t} t \, dt + \int_{0}^{10} e^{-t} \, dt
$$

$$
= - e^{-t}(t + 1) \bigg|_{0}^{10} - e^{-t} \bigg|_{0}^{10} = 1 - 11e^{-10} + 1 - e^{-10} = 2 - 12e^{-10}.
$$
Theorem 9
Let $X$ be a discrete r.v. whose possible values are nonnegative integers. Let $h : [0, \infty) \rightarrow [0, \infty)$ be a function. Let $H(x) = \int_0^x h(t) \, dt$. Then,

$$E[H(X)] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(H(k) - H(k - 1)).$$

Proof: We have that $s(t) = \mathbb{P}\{X \geq k\}$, for $k - 1 \leq t < k$. Hence,

$$E[H(X)] = \int_0^{\infty} s(t) h(t) \, dt = \sum_{k=1}^{\infty} \int_{k-1}^{k} s(t) h(t) \, dt$$
$$= \sum_{k=1}^{\infty} \int_{k-1}^{k} \mathbb{P}\{X \geq k\} h(t) \, dt = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} \int_{k-1}^{k} h(t) \, dt$$
$$= \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(H(k) - H(k - 1)).$$
Recall that if \( H(x) = \int_0^x h(t) \, dt \), then,

\[
E[H(X)] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} (H(k) - H(k - 1)).
\]

This implies that

\[
E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\},
\]

\[
E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} (k^2 - (k - 1)^2) = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\} (2k - 1)
\]

and

\[
E[\min(X, a)] = \sum_{k=1}^{a} \mathbb{P}\{X \geq k\},
\]

where \( a \) is a positive integer.
Example 8

Let $X$ be a discrete r.v. with probability mass function given by the following table,

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\mathbb{P}{X = k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

(i) Find $E[X]$ and $E[X^2]$, using that

$$E[H(X)] = \sum_{k=0}^{\infty} H(k)\mathbb{P}\{X = k\}.$$ 

(ii) Find $E[X]$ and $E[X^2]$, using that $E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}$ and $E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(2k - 1)$. 
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(i) Find $E[X]$ and $E[X^2]$, using that

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(ii) Find $E[X]$ and $E[X^2]$, using that $E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}$ and $E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(2k - 1)$.

**Solution:** (i) We have that

$$E[X] = (0)(0.2) + (1)(0.3) + (2)(0.5) = 1.3$$

$$E[X^2] = (0)^2(0.2) + (1)^2(0.3) + (2)^2(0.5) = 2.3.$$
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Let $X$ be a discrete r.v. with probability mass function given by the following table,

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<tbody>
<tr>
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<td>0.5</td>
</tr>
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(ii) Find $E[X]$ and $E[X^2]$, using that $E[X] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}$ and $E[X^2] = \sum_{k=1}^{\infty} \mathbb{P}\{X \geq k\}(2k - 1)$.

**Solution:** (ii) We have that $\mathbb{P}\{X \geq 1\} = 0.8$, $\mathbb{P}\{X \geq 2\} = 0.5$, and $\mathbb{P}\{X \geq k\} = 0$, for each $k \geq 3$. Hence,

$E[X] = \mathbb{P}\{X \geq 1\} + \mathbb{P}\{X \geq 2\} = 0.8 + 0.5 = 1.3$

$E[X^2] = \mathbb{P}\{X \geq 1\}((2)(1) - 1) + \mathbb{P}\{X \geq 2\}((2)(2) - 1)$

$= 0.8 + 0.5(3) = 2.3.$
Definition 9

Given $0 < p < 1$, the $100p$–th percentile (or $p$–th quantile) of a r.v. $X$ is a value such that

$$\mathbb{P}\{X < \xi_p\} \leq p \leq \mathbb{P}\{X \leq \xi_p\}.$$  

Usually $\mathbb{P}\{X \leq \xi_p\} = p$. If $X$ has a continuous distribution, then $\mathbb{P}\{X < \xi_p\} = \mathbb{P}\{X \leq \xi_p\}$ and $\mathbb{P}\{X \leq \xi_p\} = p$. 
Theorem 10

If $X$ has a uniform distribution on the interval $(a, b)$, then the $p$–th quantile $\xi_p$ of $X$ is $a + (b - a)p$. 
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If $X$ has a uniform distribution on the interval $(a, b)$, then the $p$–th quantile $\xi_p$ of $X$ is $a + (b - a)p$.

**Proof:** We have that

$$p = \mathbb{P}\{X \leq \xi_p\} = \int_a^{\xi_p} \frac{1}{b - a} + dt = \frac{\xi_p - a}{b - a}.$$

So, $\xi_p = a + (b - a)p$. 

Definition 10

A **median** $m$ of a r.v. $X$ is a value such that
\[ P\{X < m\} \leq \frac{1}{2} \leq P\{X \leq m\} . \]

Definition 11

The **first quartile** $Q_1$ of a r.v. $X$ is the 25–th percentile of the r.v. $X$. The **third quartile** $Q_3$ of a r.v. $X$ is the 75–th percentile of the r.v. $X$.

Usually, the range of a r.v. $X$ is divided in four parts with probability 0.25 each by the numbers $-\infty, Q_1, m, Q_3, \infty$.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$(-\infty, Q_1)$</th>
<th>$(Q_1, m)$</th>
<th>$(m, Q_3)$</th>
<th>$(Q_3, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>25 %</td>
<td>25%</td>
<td>25%</td>
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</tr>
</tbody>
</table>
Example 9

Suppose that the age–at–failure r.v. $X$ has density

$$f_X(x) = \begin{cases} \frac{5x^4}{k^5} & \text{if } 0 < x < k, \\ 0 & \text{else.} \end{cases}$$

Suppose that the expected age–at–failure is 70 years. Find the median age–at–failure.
Example 9

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*Suppose that the expected age–at–failure is 70 years. Find the median age–at–failure.*

**Solution:** Since

$$70 = E[X] = \int_0^k x \frac{5x^4}{k^5} \, dx = \left[ \frac{5x^6}{6k^5} \right]_0^k = \frac{5k}{6},$$

$$k = \frac{(70)(6)}{5} = 84.$$ Let $m$ be the median age–at–failure. Then,

$$\frac{1}{2} = \int_0^m \frac{5x^4}{(84)^5} \, dx = \left[ \frac{x^5}{(84)^5} \right]_0^m = \frac{m^5}{(84)^5},$$

and $m = \frac{84}{2^{5/5}} = 73.12624732.$
Theorem 11

Let $X$ be a continuous r.v. with density function $f_X$. Let $0 < p < 1$. Suppose that there are $-\infty \leq a < b \leq \infty$ such that:

(i) $f_X(x) = 0$, if $x \not\in (a, b)$.

(ii) $f_X(x)$ is continuous and positive in $(a, b)$.

Then, there exists $\xi_p$ such that $F_X(\xi_p) = p$. Moreover, $\xi_p$ is unique.
Theorem 12

Let $X$ be a r.v. with range $(a, b)$ and density $f_X$. Let $0 < p < 1$. Let $h : (a, b) \to (c, d)$ be a one–to–one onto function.

(i) Let $\xi_p$ be a $p$–th quantile of $X$ such that
$$P\{X < \xi_p\} = p = P\{X \leq \xi_p\}.$$ If $h$ is nonincreasing, then a $p$–the quantile of $Y$ is $\zeta_p = h(\xi_p)$.

(ii) Let $\xi_{1-p}$ be a $(1 - p)$–th quantile of $X$ such that
$$P\{X < \xi_{1-p}\} = 1 - p = P\{X \leq \xi_{1-p}\}.$$ If $h$ is nondecreasing, then a $p$–the quantile of $Y$ is $\zeta_p = h(\xi_{1-p})$. 
Example 10

Suppose that the age–at–failure r.v. $X$ has density

$$f_X(x) = \begin{cases} \frac{5x^4}{(84)^5} & \text{if } 0 < x < 84, \\ 0 & \text{else.} \end{cases}$$

Find the three quartiles of $(1000)(1.06)^{-X}$. 

Example 10

Suppose that the age–at–failure r.v. $X$ has density

$$f_X(x) = \begin{cases} \frac{5x^4}{(84)^5} & \text{if } 0 < x < 84, \\ 0 & \text{else.} \end{cases}$$

Find the three quartiles of $(1000)(1.06)^{-X}$.

**Solution:** (i) let $h(x) = (1000)(1.06)^{-x}$, $x \geq 0$. $h$ is a decreasing function. Let $\xi_p$ be $p$–th quantile of the r.v. $X$. Let $\zeta_p$ be $p$–th quantile of the r.v. $h(X)$. By the previous theorem, $\zeta_p = h(\xi_{1-p})$.

Hence,

$$\zeta_{0.25} = h(\xi_{0.75}) = (1000)(1.06)^{-79.30335095} = 9.843738901,$$

$$\zeta_{0.5} = h(\xi_{0.5}) = (1000)(1.06)^{-73.12624732} = 14.10837641,$$

$$\zeta_{0.75} = h(\xi_{0.25}) = (1000)(1.06)^{-63.6609579} = 24.49210954,$$