Manual for SOA Exam MLC.

Chapter 2. Survival models.

Section 2.2. Actuarial notation for survival analysis..

Extract from:

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Future lifetime of a life aged x.

Definition 1

We denote by (x) to a life that survives to age x.

(x) is called a **life–age–**x or a life aged x.

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Definition 2

We denote by $T(x) = X - x$ *is the* **future lifetime** *of* (x)*, i.e.* $T(x) = X - x$ *is* **time–until–death** *of an entity with age* x*.*

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Definition 2

We denote by $T(x) = X - x$ is the **future lifetime** *of (x), i.e. $T(x) = X - x$ is* **time–until–death** *of an entity with age x.*

An alternative notation for $T(x)$ is T_x. Notice that $T_0 = X$, i.e. the future lifetime of an entity with age zero is the age–at–failure. Notice that since $T(x)$ is defined only for a life which survives to age x, probabilities for $T(x)$ are conditional probabilities.

The survival function of $T(x)$ is

$$\mathbb{P}\{T(x) > t\} = \mathbb{P}\{X - x > t | X > x\} = \frac{\mathbb{P}\{X - x > t, X > x\}}{\mathbb{P}\{X > x\}}$$

$$= \frac{s(x+t)}{s(x)}, t \geq 0.$$

The density of $T(x)$ is

$$f_{T(x)}(t) = -\frac{d}{dt}\mathbb{P}\{T(x) > t\} = -\frac{d}{dt}\frac{s(x+t)}{s(x)} = \frac{f_X(x+t)}{s(x)}, t \geq 0.$$

We denote the density of $T(x)$ by

$$f_X(x+t|X > x) = \frac{f_X(x+t)}{s(x)}, t \geq 0.$$

The cumulative distribution function of $T(x)$ is

$$\mathbb{P}\{T(x) \leq t\} = 1 - \mathbb{P}\{T(x) > t\} = \frac{s(x) - s(x+t)}{s(x)}.$$

Example 1

Consider the survival function $S_X(t) = \frac{90^6 - t^6}{90^6}$, *for* $0 < t < 90$. *Find the survival function and the probability density function of* $T(x)$.

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Solution: The survival function of $T(x)$ is

$$\frac{s(x+t)}{s(x)} = \frac{\frac{90^6-(x+t)^6}{90^6}}{\frac{90^6-x^6}{90^6}} = \frac{90^6 - (x+t)^6}{90^6 - x^6}, 0 \le t \le 90 - x.$$

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The probability density function of $T(x)$ is

$$f_{T(x)}(t) = -\frac{d}{dt}\frac{s(x+t)}{s(x)} = -\frac{d}{dt}\left(\frac{90^6-(x+t)^6}{90^6-x^6}\right)$$

$$= \frac{6(x+t)^5}{90^6-x^6}, 0 \le t \le 90 - x.$$

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Notice that ${}_tp_x$ is the survival function of $T(x)$, i.e.

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The survival function of the age–at–death X is ${}_tp_0$, $t \geq 0$.

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The survival function of the age–at–death X is ${}_tp_0$, $t \geq 0$.
Since ${}_tp_x$, $t \geq 0$, is the survival function of $T(x)$,

$$ f_{T(x)}(t) = -\frac{d}{dt}{}_tp_x, t \geq 0, $$

and

$$ {}_tp_x = \int_t^\infty f_{T(x)}(t)\, ds, t \geq 0. $$

Example 2

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Solution: The probability that a 25–year-old reaches age 80 is

$$ {}_{80-25}p_{25} = {}_{55}p_{25} = 1 - \frac{55}{90-25} = \frac{2}{13}. $$

Example 3

Suppose that ${}_tp_x = 1 - \frac{t}{90-x}$, $0 \le t \le 90 - x$, *find the density of* $T(x)$.

Solution: The density of $T(x)$.

$$f_{T(x)}(t) = -\frac{d}{dt}\,{}_tp_x = \frac{1}{90-x}, 0 \le t \le 90 - x.$$

Theorem 1

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Proof: We have that

$$ {}_m p_x \cdot {}_n p_{x+m} = \frac{s(x+m)}{s(x)} \frac{s(x+m+n)}{s(x+m)} = \frac{s(x+m+n)}{s(x)} = {}_{m+n}p_x. $$

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By induction, the previous theorem implies that

$$ {}_{\sum_{j=1}^k n_j} p_x = {}_{n_1}p_x \cdot {}_{n_2}p_{x+n_1} \cdot {}_{n_3}p_{x+n_1+n_2} \cdots {}_{n_k}p_{x+\sum_{j=1}^{k-1} n_j}. $$

Example 4

Suppose that probability that a 30–year–old reaches age 40 is 0.95*, the probability that a 40–year–old reaches age 50 is* 0.90*, and the probability that a 50–year–old reaches age 60 is* 0.95*. Find the probability that a 30–year–old reaches age 60.*

Example 4

Suppose that probability that a 30–year–old reaches age 40 is 0.95*, the probability that a 40–year–old reaches age 50 is* 0.90*, and the probability that a 50–year–old reaches age 60 is* 0.95*. Find the probability that a 30–year–old reaches age 60.*

Solution: The probability that a 30–year–old reaches age 60 is

$$ {}_{30}p_{30} = {}_{10}p_{30} \cdot {}_{10}p_{40} \cdot {}_{10}p_{50} = (0.95)(0.99)(0.95) = 0.893475. $$

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${}_tq_0$ is the cumulative distribution function of the age–at–death X.

Example 5

Suppose that ${}_tq_x = \frac{t}{95-x}$, $0 \le t \le 95 - x$, *find the probability that a 30–year–old dies before age 75.*

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Solution: The probability that a 30–year–old dies before age 75 is

$$ {}_{75-30}q_{30} = {}_{45}q_{30} = \frac{45}{95-30} = \frac{9}{13}. $$

Definition 5

p_x is the probability that a life aged x survives one year, i.e.

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Notice that $p_x = {}_1p_x$.

Theorem 2

$$ {}_{n}p_x = p_x p_{x+1} \cdots p_{x+n-1}. $$

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$$_np_x = p_x p_{x+1} \cdots p_{x+n-1}.$$

Proof: We have that

$$p_x p_{x+1} \cdots p_{x+n-1} = \frac{s(x+1)}{s(x)} \frac{s(x+2)}{s(x+1)} \cdots \frac{s(x+n)}{s(x+n-1)}$$

$$= \frac{s(x+n)}{s(x)} = {}_np_x.$$

Example 6

Suppose that

k	29	30	31	32	33	34	35	36
p_k	0.99	0.98	0.97	0.96	0.96	0.95	0.94	0.93

Find the probability that a 30–year old survives to age 35.

Example 6

Suppose that

k	29	30	31	32	33	34	35	36
p_k	0.99	0.98	0.97	0.96	0.96	0.95	0.94	0.93

Find the probability that a 30–year old survives to age 35.

Solution: The probability that a 30–year old survives to age 35 is

$$ {}_5p_{30} = p_{30}p_{31}p_{32}p_{33}p_{34} = (0.98)(0.97)(0.96)(0.96)(0.95) $$
$$ = 0.832269312. $$

Definition 6

q_x is the probability that a life aged x will die within one year, i.e.

$$q_x = \mathbb{P}\{X \leq x + 1 | X > x\} = \frac{s(x) - s(x+1)}{s(x)}.$$

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Notice that $q_x = {}_1q_x$.

Example 7

Consider the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$.
Find ${}_tp_x,\ {}_tq_x, p_x, q_x$.

Example 7

Consider the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find* ${}_t p_x$, ${}_t q_x$, p_x, q_x.

Solution: We have that

$$ {}_t p_x = \frac{s(x+t)}{s(x)} = \frac{\frac{90^6-(x+t)^6}{90^6}}{\frac{90^6-x^6}{90^6}} = \frac{90^6-(x+t)^6}{90^6-x^6}, 0 \le t \le 90-x, $$

$$ {}_t q_x = \frac{s(x)-s(x+t)}{s(x)} = \frac{\frac{90^6-x^6}{90^6} - \frac{90^6-(x+t)^6}{90^6}}{\frac{90^6-x^6}{90^6}} $$

$$ = \frac{(x+t)^6-x^6}{90^6-x^6}, 0 \le t \le 90-x, $$

$$ p_x = {}_1 p_x = \frac{90^6-(x+1)^6}{90^6-x^6}, 0 \le x \le 89, $$

$$ q_x = {}_1 q_x = \frac{(x+1)^6-x^6}{90^6-x^6}, 0 \le x \le 89. $$

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$$1 = q_k + p_k q_{k+1} + {}_2p_k q_{k+2} + {}_3p_k q_{k+3} + \cdots$$

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Proof: We have that $q_k = \mathbb{P}\{X \le k\}$,

$$p_k q_{k+1} = \frac{s(k) - s(k+1)}{s(k)} s(k) = s(k) - s(k+1) = \mathbb{P}\{k < X \le k+1\},$$

$$\begin{aligned} {}_2p_k q_{k+2} &= \frac{s(k+2)}{s(k)} \frac{s(k+1) - s(k+2)}{s(k+2)}) \\ &= s(k+1) - s(k+2) = \mathbb{P}\{k+1 < X \le k+2\} \end{aligned}$$

and so on. Hence,

$$\begin{aligned} & q_k + p_k q_{k+1} + {}_2p_k q_{k+2} + {}_3p_k q_{k+3} + \cdots \\ =& \mathbb{P}\{X \le k\} + \mathbb{P}\{k < X \le k+1\} + \mathbb{P}\{k+1 < X \le k+2\} + \cdots = 1. \end{aligned}$$

Definition 7

Given $x, s, t > 0$, ${}_{s|t}q_x$ represents the probability of a life just turning age x will die between ages $x + s$ and $x + s + t$, i.e.

$$ {}_{s|t}q_x = \mathbb{P}\{s < T(x) \leq s + t\} = \frac{s(x+s) - s(x+s+t)}{s(x)}. $$

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The symbol ${}_{s|t}$ means deferred for s years and happening within the next m years.

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The symbol ${}_{s|t}$ means deferred for s years and happening within the next m years.
We abbreviate ${}_{t|}q_x = {}_{t|t+1}q_x$.

Example 8

Consider the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$.
(i) Find ${}_{s|t}q_x$, *for* $0 < x, s, t$ *and* $x + s + t \leq 90$.
(ii) Find the probability that a 30-year-old dies between ages 55 and 60.

Example 8

Consider the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$.
(i) Find ${}_{s|t}q_x$, *for* $0 < x, s, t$ *and* $x + s + t \leq 90$.
(ii) Find the probability that a 30-year-old dies between ages 55 and 60.

Solution: (i) We have that

$$ {}_{s|t}q_x = \frac{s(x+s) - s(x+s+t)}{s(x)} = \frac{\frac{90^6-(x+s)^6}{90^6} - \frac{90^6-(x+s+t)^6}{90^6}}{\frac{90^6-x^6}{90^6}} $$

$$ = \frac{(x+s+t)^6 - (x+s)^6}{90^6 - x^6}. $$

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(i) Find ${}_{s|t}q_x$, *for* $0 < x, s, t$ *and* $x + s + t \leq 90$.
(ii) Find the probability that a 30-year-old dies between ages 55 and 60.

Solution: (i) We have that

$$ {}_{s|t}q_x = \frac{s(x+s) - s(x+s+t)}{s(x)} = \frac{\frac{90^6-(x+s)^6}{90^6} - \frac{90^6-(x+s+t)^6}{90^6}}{\frac{90^6-x^6}{90^6}} $$

$$ = \frac{(x+s+t)^6 - (x+s)^6}{90^6 - x^6}. $$

(ii) The probability that a 30–year–old dies between ages 55 and 60 is

$$ {}_{25|5}q_{30} = \frac{(60)^6 - (55)^6}{(90)^6 - (30)^6} = 0.03575453. $$

Theorem 4

$$ {}_{s|t}q_x = {}_s p_x - {}_{s+t}p_x = {}_s p_x \cdot {}_t q_{x+s}. $$

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$$ {}_{s|t}q_x = {}_sp_x - {}_{s+t}p_x = {}_sp_x \cdot {}_tq_{x+s}. $$

Proof: We have that

$$ {}_{s|t}q_x = \mathbb{P}\{s < T(x) \le s+t\} = \mathbb{P}\{s < T(x)\} - \mathbb{P}\{s+t < T(x)\} $$
$$ = {}_sp_x - {}_{s+t}p_x $$

and

$$ {}_sp_x \cdot {}_tq_{x+s} = \frac{s(x+s)}{s(x)} \frac{s(x+s) - s(x+s+t)}{s(x+s)} $$
$$ = \frac{s(x+s) - s(x+s+t)}{s(x)} = {}_{s|t}q_x. $$

Example 9

Suppose that:
(i) The probability that a 30–year–old will reach age 60 is 0.90.
(ii) The probability that a 30–year–old will reach age 50 is 0.95.
Find the probability that a 30–year–old will die between age 50 and 60.

Example 9

Suppose that:
(i) The probability that a 30–year–old will reach age 60 is 0.90.
(ii) The probability that a 30–year–old will reach age 50 is 0.95.
Find the probability that a 30–year–old will die between age 50 and 60.

Solution: We have that

$$ {}_{20|10}q_{30} = {}_{20}p_{30} - {}_{30}p_{30} = 0.95 - 0.90 = 0.05. $$

Example 10

Suppose that:
(i) The probability that a 30–year–old will reach age 50 is 0.90.
(ii) The probability that a 50–year–old will reach age 60 is 0.95.
Find the probability that a 30–year–old will die between age 50 and 60.

Example 10

Suppose that:
(i) The probability that a 30–year–old will reach age 50 is 0.90.
(ii) The probability that a 50–year–old will reach age 60 is 0.95.
Find the probability that a 30–year–old will die between age 50 and 60.

Solution: We have that

$$_{20|10}q_{30} = {}_{20}p_{30} \cdot {}_{10}q_{50} = (0.90)(1 - 0.95) = 0.045.$$

